THE UNIVERSITY OF ALBERTA

STUDIES ON THE STABILITY OF TIME-VARYING SYSTEMS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled STUDIES ON THE STABILITY OF TIME-VARYING SYSTEMS submitted by BERNHARD BUSSMANN in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

The stability of systems is investigated which can be described by differential equations consisting of a linear time-varying part and one nonlinear term.

First, only the linear part is considered. Definition and properties of the impulse response are discussed in detail. Closed form solutions are evaluated for some special systems. For the general case, approximations are determined by comparing two integral equations.

Then, utilizing the results of the linear part, an integral equation is set up for the complete system, including the nonlinear term. Sufficient stability conditions are derived. It is shown that the appropriate Volterra series is convergent under relatively weak restrictions.

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CHAPTER I

INTRODUCTION

1.1 Preliminary Remarks

Continuous systems with lumped parameters in general can be described by ordinary time-varying differential equations. Since it is difficult or often impossible to solve these equations, the investigation of a physical system usually requires admissible assumptions which simplify the mathematical description. Fig. 1.A illustrates a customary procedure for linearizing a system or to make it time-invariant (stationary).

For a linear time-invariant system, a complete mathematical theory is available which has been successfully applied to numerous problems. However, in those cases where the results, obtained from the linearized time-invariant system, are insufficient or the system does not allow such a simplification, nonlinearities and (or) time-varying parameters have to be included.

In spite of the huge amount of work which has been done in the field of nonlinear time-varying differential equations, the results do not form a complete theory yet. This is mainly due to the fact that solutions, in closed form, exist only for a very few special classes of systems. Consequently, for the general case, one has to resort to various approximations depending on the particular problem and the type of system under consideration. Moreover, it is often preferred not to

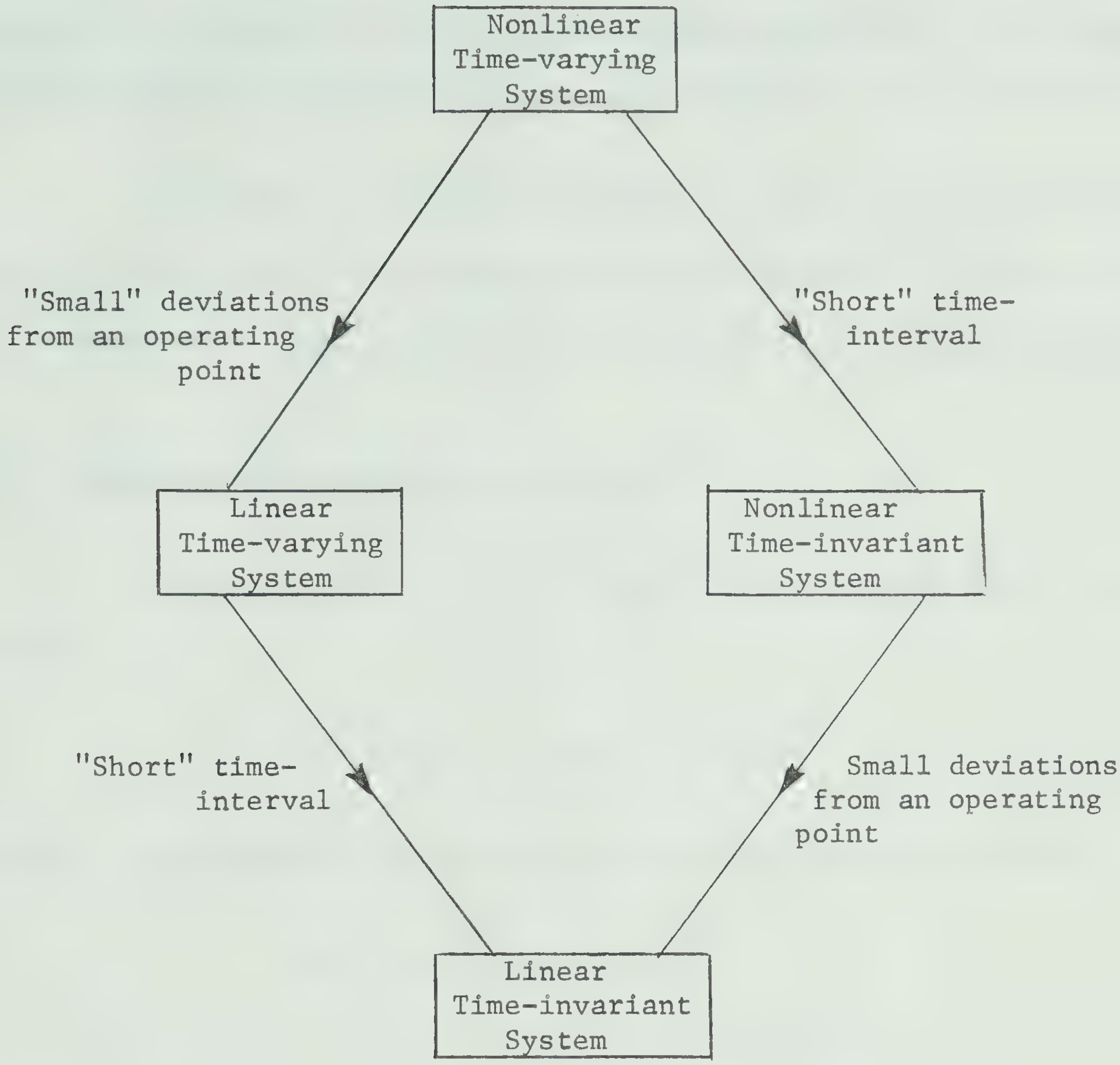

Nonlinear Time-varying System
"Small" deviations from an operating point
"Short" time-interval
Linear Time-varying System
Nonlinear Time-invariant System
"Short" time-interval
Small deviations from an operating point
Linear Time-invariant System

Figure 1.A

determine the complete solution, but to investigate only those properties which have to be known for a certain application, for example: zeros, extrema, monotonic or oscillatory behavior, envelopes, etc.

From these introductory remarks it follows that the objective of this thesis has to be limited to the investigation of some properties of a class of systems, which will be outlined in the following section.

1.2 Objective and Outline of the Method

In this thesis, the following class of systems will be dealt with:

$$L(p,t)\ x(t) + \varepsilon\ \phi(x,t) = r(t)\ . \tag{1.2.1}$$

$L(p,t)$ is a nonsingular, time-varying linear operator defined as:

$$L(p,t) = p^n + a_{n-1}(t)\ p^{n-1} + \dots \\ \dots + a_1(t)\ p + a_o(t) \tag{1.2.2}$$

where:

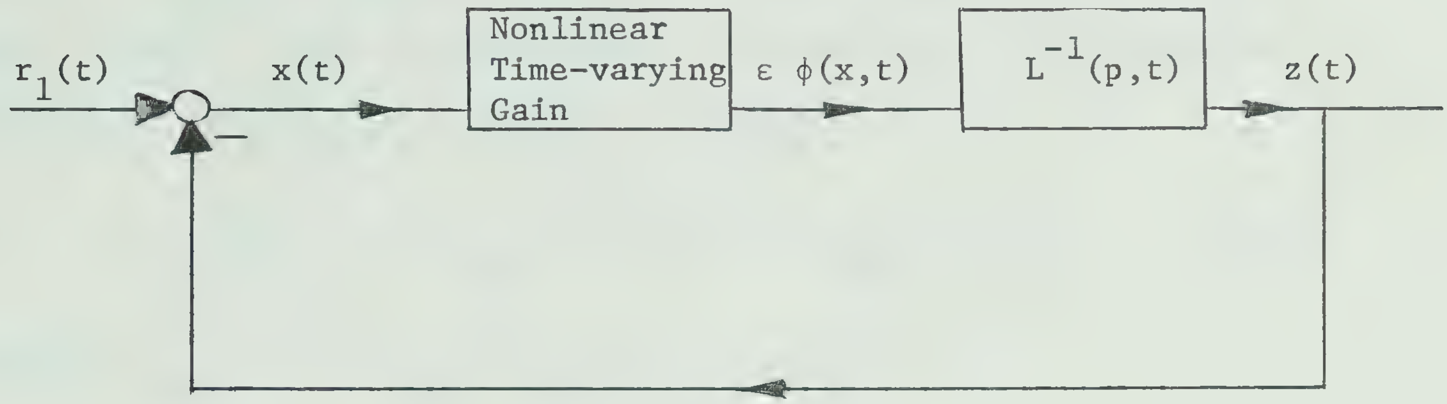
$$p^k = \frac{d^k}{dt^k} \quad ,$$

$r(t)$ is an arbitrary driving function,

ε is a positive constant, and

$\phi(x,t)$ is a nonlinear function, specified in detail in Chapter IV.

Equation (1.2.1) describes, for instance, the feedback systems in Fig. 1.B and Fig. 1.C.



The function $r(t)$ in (1.2.1) is defined as:

$$r(t) = L(p,t)\ r_1(t)\ .$$

Figure 1.B

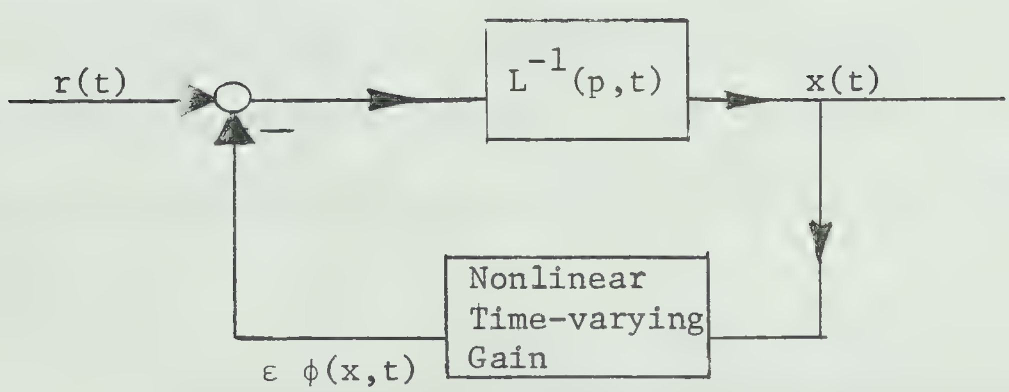


Figure 1.C

System (1.2.1) becomes more familiar, when the special case $L(p,t) = L(p)$ is considered, in other words, in the case of the stationary linear differential operator:

$$L(p) = p^n + a_{cn-1}\, p^{n-1} + \ldots + a_{c1}\, p + a_{co} \qquad (1.2.3)$$

where $a_{cn-1}, \ldots, a_{c1}, a_{co}$ are constants.

Then (1.2.1) becomes:

$$L(p)\, x(t) + \varepsilon\, \phi(x,t) = r(t)\ . \qquad (1.2.4)$$

One of the properties of (1.2.4), which has been the object of extensive research, is the stability for various inputs or initial conditions. The two best known results are the Popov and circle criteria, which both use a frequency domain approach to derive sufficient stability conditions. Brockett [1], Barrett [2], Trott and Christensen [3], and others apply certain results of functional analysis to equation (1.2.4) in order to get sufficient stability criteria and to generate a series of successive approximations.

A prerequisite for applying these methods is the determination of the inverse operator $L^{-1}(p)$, so that the solution of (1.2.4) can be written as:

$$x(t) = L^{-1}(p)\, \{r(t) - \varepsilon\, \phi(x,t)\} \qquad (1.2.5)$$

or in form of the convolution integral:

$$x(t) = y(t) + \int_0^t h(t-\tau)[r(\tau) - \varepsilon\ \phi(x,\tau)]\ d\tau$$

where:

$y(t)$ is the solution of

$$L(p)\ y(t) = 0$$

subject to the given initial conditions, and $h(t-\tau)$ is the so called impulse response.

For the time-invariant operator (1.2.3), the functions $y(t)$ and $h(t-\tau)$ can be obtained in closed form.

If it is desired to use a similar approach for the investigation of the time-varying system (1.2.1), one could formally rewrite (1.2.1) as:

$$x(t) = L^{-1}(p,t)\ \{r(t) - \varepsilon\ \phi(x,t)\} \qquad (1.2.7)$$

or in integral form:

$$x(t) = y(t) + \int_0^t h(t,\tau)\ [r(\tau) - \varepsilon\ \phi(x,\tau)]\ d\tau\ . \qquad (1.2.8)$$

The major part of this thesis will deal with the problem of justifying (1.2.8) and finding closed form solutions and approximations for $y(t)$ and $h(t,\tau)$.

In Chapter II, basic properties of linear time-varying differential equations are reviewed which are relevant to the impulse

response $h(t,\tau)$. Several classes of systems, which permit solutions in closed form, illustrate the properties of $h(t,\tau)$ and the manner in which it is computed.

In Chapter III, a unified method is presented for approximating certain types of linear systems.

In Chapter IV, the results of Chapter II and III are applied to (1.2.1). Sufficient stability conditions are derived and the convergence of the Volterra series, developed from (1.2.8), is proved.

CHAPTER II

LINEAR TIME-VARYING DIFFERENTIAL EQUATIONS

2.1 Introduction

This chapter deals with the solution of linear time-varying differential equations.

In section 2.2, basic properties of linear differential equations are reviewed, which are needed in order to derive $h(t,\tau)$ as the one-sided Green's function. In this manner, one arrives at a simple and rigorous definition for $h(t,\tau)$.

In section 2.3, $h(t,\tau)$ is evaluated in closed form for some special systems. A transformation in the time domain is introduced which reduces some time-varying systems to systems with constant coefficients.

2.2 Review of Basic Properties [4]

Consider the following linear differential equation:

$$L(p,t)\, x(t) = \{p^n + a_{n-1}(t)\, p^{n-1} + \cdots \qquad (2.2.1)$$
$$\cdots + a_1(t)\, p + a_o(t)\}\, x(t) = r(t)$$

with

$$p^k = \frac{d^k}{dt^k}$$

subject to the initial conditions:

$$p^{n-1}x\Big|_{t=t_o} = x_o^{(n-1)} \ , \ p^{n-2}x\Big|_{t=t_o} = x_o^{(n-2)} \ , \ \cdots$$
$$\cdots \ , \ px\Big|_{t=t_o} = \dot{x}_o \ , \ x(t_o) = x_o \ . \tag{2.2.2}$$

The solution of (2.2.1) is the sum of the homogeneous and the forced solution:

$$x(t) = y(t) + z(t) \quad . \tag{2.2.3}$$

The homogeneous solution $y(t)$ satisfies:

$$L(p,t)\; y(t) = 0 \tag{2.2.4}$$

and includes the initial conditions (2.2.2), that is:

$$y^{(n-1)}(t_o) = x_o^{(n-1)} \ , \ \cdots \ , \ y(t_o) = x_o \ , \tag{2.2.5}$$

Existence and uniqueness of $y(t)$ are guaranteed by the Cauchy-Lipschitz condition which, in the case of (2.2.1), is satisfied if the coefficients $a_k(t)$ are continuous functions of t. Throughout this thesis it will be assumed that the coefficients $a_k(t)$ satisfy the more restrictive condition to have derivatives of any desirable order.

The homogeneous solution $y(t)$ can be expressed as a linear combination of n linearly independent functions $\phi_k(t)$, each of which satisfies (2.2.4).

Thus:

$$y(t) = c_1 \phi_1(t) + \cdots + c_n \phi_n(t) .$$

Linear independence implies that the constants c_k can be determined from the initial conditions (2.2.5).

The forced solution $z(t)$ is defined by:

$$L(p,t)\; z(t) = r(t) \tag{2.2.7}$$

for the following initial conditions:

$$z^{(n-1)}(t_o) = z^{(n-2)}(t_o) = \cdots = \dot{z}(t_o) = z(t_o) = 0 . \tag{2.2.8}$$

The solution of (2.2.7) is conveniently written as convolution integral:

$$z(t) = \int_{t_o}^{t} h(t,\tau)\; r(\tau)\; d\tau \quad . \tag{2.2.9}$$

A simple and rigorous procedure to derive (2.2.9) and to define $h(t,\tau)$ uses the method of 'Variation of Parameters'. This way avoids in addition the lengthy introduction of a mathematically acceptable δ-function [5]. In order to illustrate this procedure in some detail, consider an example of second order:

$$\ddot{z}(t) + a_1(t)\,\dot{z}(t) + a_o(t)\,z(t) = r(t) \tag{2.2.10}$$

with the initial conditions:

$$z(t_o) = 0 \quad \text{and} \quad \dot{z}(t_o) = 0 \tag{2.2.11}$$

Let $\phi_1(t)$ and $\phi_2(t)$ be two linearly independent solutions satisfying the homogeneous solution:

$$\ddot{\phi}_{1,2} + a_1(t)\,\dot{\phi}_{1,2} + a_o(t)\,\phi_{1,2} = 0$$

Then the method of 'Variation of Parameters' tries to determine two functions $\xi_1(t)$ and $\xi_2(t)$ such that the solution of (2.2.10) can be written as:

$$z(t) = \xi_1(t)\,\phi_1(t) + \xi_2(t)\,\phi_2(t) \tag{2.2.12}$$

Differentiating (2.2.12) gives:

$$\dot{z}(t) = \dot{\xi}_1\,\phi_1 + \dot{\xi}_2\,\phi_2 + \xi_1\,\dot{\phi}_1 + \xi_2\,\dot{\phi}_2 \tag{2.2.13}$$

The first equation chosen for the unknown functions ξ_1 and ξ_2 is:

$$\dot{\xi}_1\,\phi_1 + \dot{\xi}_2\,\phi_2 = 0 \tag{2.2.14}$$

Consequently, the second derivative of $z(t)$ is:

$$\ddot{z} = \xi_1\,\ddot{\phi}_1 + \xi_2\,\ddot{\phi}_2 + \dot{\xi}_1\,\dot{\phi}_1 + \dot{\xi}_2\,\dot{\phi}_2 \tag{2.2.15}$$

The equations (2.2.12), (2.2.13), and (2.2.15) have to satisfy (2.2.10). This yields the second equation for $\xi_1(t)$ and $\xi_2(t)$:

$$\begin{aligned} \xi_1[\ddot{\phi}_1 + a_1(t)\,\dot{\phi}_1 + a_o(t)\,\phi_1] + \xi_2[\ddot{\phi}_2 + a_1(t)\,\dot{\phi}_2 + \\ + a_o(t)\,\phi_2] + \dot{\xi}_1\,\dot{\phi}_1 + \dot{\xi}_2\,\dot{\phi}_2 = r(t) \end{aligned} \tag{2.2.16}$$

Equations (2.2.14) and (2.2.16)

$$\dot{\xi}_1\,\phi_1 + \dot{\xi}_2\,\phi_2 = 0$$

$$\dot{\xi}_1\,\dot{\phi}_1 + \dot{\xi}_2\,\dot{\phi}_2 = r(t)$$

can be solved for $\dot{\xi}_1$ and $\dot{\xi}_2$, so that:

$$\begin{aligned} D(t)\,\dot{\xi}_1 &= -\,r(t)\,\phi_2(t)\ , \\ D(t)\,\dot{\xi}_2 &= +\,r(t)\,\phi_1(t) \end{aligned} \tag{2.2.17}$$

with

$$D(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{vmatrix} \neq 0\ . \tag{2.2.18}$$

Integrating (2.2.17)

$$\begin{aligned} \xi_1(t) &= \xi_1(t_o) - \int_{t_o}^{t} \frac{\phi_2(\tau)}{D(\tau)}\,r(\tau)\,d\tau \\ \xi_2(t) &= \xi_2(t_o) + \int_{t_o}^{t} \frac{\phi_1(\tau)}{D(\tau)}\,r(\tau)\,d\tau \end{aligned} \tag{2.2.19}$$

and substituting (2.2.19) into (2.2.12) gives:

$$z(t) = \xi_1(t_o)\ \phi_1(t) + \xi_2(t_o)\ \phi_2(t)$$

$$- \int_{t_o}^{t} \frac{\phi_1(t)\ \phi_2(\tau) - \phi_2(t)\ \phi_1(\tau)}{D(\tau)}\ r(\tau)\ d\tau \qquad (2.2.20)$$

Observing the initial conditions (2.2.11) results in:

$$\xi_1(t_o) = 0 \quad \text{and} \quad \xi_2(t_o) = 0$$

so that (2.2.20) becomes:

$$z(t) = \int_{t_o}^{t} - \frac{\phi_1(t)\ \phi_2(\tau) - \phi_2(t)\ \phi_1(\tau)}{D(\tau)}\ r(\tau)\ d\tau \qquad (2.2.21)$$

Comparing (2.2.21) with (2.2.9) yields the definition of $h(t,\tau)$:

$$h(t,\tau) = - \frac{\phi_1(t)\ \phi_2(\tau) - \phi_2(t)\ \phi_1(\tau)}{D(\tau)}$$

$$= - \frac{\begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1(\tau) & \phi_2(\tau) \end{vmatrix}}{\begin{vmatrix} \phi_1(\tau) & \phi_2(\tau) \\ \dot{\phi}_1(\tau) & \dot{\phi}_2(\tau) \end{vmatrix}} \qquad (2.2.22)$$

It is obvious that $h(t,\tau)$ is a solution of: (note: $p = \frac{d}{dt}$)

$$p^2\ h(t,\tau) + a_1(t)\ p\ h(t,\tau) + a_o(t)\ h(t,\tau) = 0 \qquad (2.2.23)$$

for the initial conditions

$$h(\tau,\tau) = 0 \quad \text{and} \quad p\, h(t,\tau)\Big|_{t=\tau} = 1 \tag{2.2.24}$$

An immediate consequence of (2.2.23) and (2.2.24) is that the existence and uniqueness of $h(t,\tau)$ is assured, whenever (2.2.4) has a unique solution which is quaranteed by the Cauchy-Lipschitz condition.

Another property of $h(t,\tau)$ which will be of interest later is the dependence on the parameter τ. For the second order example (2.2.10) it will be shown that if: $h(t,\tau)$ is the impulse response of

$$\ddot{y} + a_1(t)\, \dot{y} + a_o(t)\, y = 0$$

and $g(t,\tau)$ is the impulse response of the adjoint equation:

$$\ddot{y}_a(t) - \frac{d}{dt}\, [a_1(t)\, y_a(t)] + a_o(t)\, y_a(t) = 0$$

which implies:

$$g(\tau,\tau) = 0 \quad \text{and} \quad p\, g(t,\tau)\Big|_{t=\tau} = 1 \tag{2.2.25}$$

then

$$h(t,\tau) = -g(\tau,t). \tag{2.2.26}$$

In order to prove (2.2.26), the following expression will be integrated by parts:

(lower case indices denoting partial derivatives)

$$\int_{\tau}^{t} g(u,t)\ [h_{uu}(u,\tau) + a_1(u)\ h_u(u,\tau) + a_o(u)\ h(u,\tau)]\ du = 0$$

$$= \int_{\tau}^{t} h(u,\tau)\ [g_{uu}(u,t) - \frac{\partial}{\partial u} \{a_1(u)\ g(u,t)\} + a_o(u)\ g(u,t)]\ du$$

$$+ a_1(u)\ h(u,\tau)\ g(u,t) \Big|_{\tau}^{t} + h_u(u,\tau)\ g(u,t) \Big|_{\tau}^{t} - h(u,\tau)\ g_u(u,t) \Big|_{\tau}^{t}$$

Observing (2.2.24) and (2.2.25) gives

$$h_u(u,\tau)\ g(u,t) \Big|_{\tau}^{t} = h(u,\tau)\ g_u(u,t) \Big|_{\tau}^{t}$$

or the desired result:

$$- g(\tau,t) = h(t,\tau)$$

If the previous derivation is extended to higher order systems, the following result is arrived at:

The impulse response or one-sided Green's function $h(t,\tau)$ of (2.2.4) is the solution of (2.2.4), that is:

$$p^n\ h(t,\tau) + a_{n-1}(t)\ p^{n-1}\ h(t,\tau) + \cdots$$

$$\cdots + a_1(t)\ p\ h(t,\tau) + a_o(t)\ h(t,\tau) = 0$$

which satisfies the following initial conditions:

$$p^k\, h(t,\tau)\Big|_{t=\tau} = 0 \quad \text{for} \quad k = 0,\ 1,\ \cdots,\ n-2$$

$$p^{n-1}\, h(t,\tau)\Big|_{t=\tau} = 1 \tag{2.2.27}$$

If τ is the independent variable, $h(t,\tau)$ satisfies the adjoint equation: (note: $p = \frac{d}{d\tau}$)

$$(-1)^n\, p^n\, h(t,\tau) + (-1)^{n-1}\, p^{n-1}\{h(t,\tau)\ a_{n-1}(\tau)\} + \cdots$$

$$\cdots - p\{h(t,\tau)\ a_1(\tau)\} + a_o(t)\ h(t,\tau) = 0 \tag{2.2.28}$$

for the following initial conditions:

$$p^k\, h(t,\tau)\Big|_{t=\tau} = 0 \quad \text{for} \quad k = 0,\ 1,\ \cdots,\ n-2$$

$$p^{n-1}\, h(t,\tau)\Big|_{t=\tau} = (-1)^{n-1}$$

For the convolution integral (2.2.9), $h(t,\tau)$ has to be determined on the interval $0 \leq \tau \leq t < \infty$.

2.3 Solutions in Closed Form

In this section, solutions in closed form will be obtained for some classes of linear homogeneous differential equations. By imposing the conditions (2.2.27), the appropriate impulse response will be obtained in each case.

a. First order system

$$\dot{y} + a_o(t)\ y(t) = 0 \tag{2.3.1}$$

Solution:

$$y(t) = y(t_o)\, e^{-\int_{t_o}^{t} a_o(u)\, du}$$

According to (2.2.27), the impulse response is the solution of (2.3.1) which satisfies:

$$y(t_o) = y(\tau) = 1$$

Consequently:

$$h(t,\tau) = e^{-\int_{\tau}^{t} a_o(u)\,du}$$

b. Second order systems

$$\ddot{y} + a_1(t)\, \dot{y} + a_o(t)\, y = 0 \tag{2.3.2}$$

In order to obtain the solution of (2.3.2) in closed form, certain assumptions have to be made regarding the coefficients $a_1(t)$ and $a_o(t)$.

1. Let $a_o(t) = \dot{a}_1(t)$

   Then (2.3.2) can be rewritten as follows:

$$\ddot{y} + \frac{d}{dt}\, [y\; a_1(t)] = 0 \tag{2.3.3}$$

   The general solution of (2.3.3) is:

$$y(t) = y(t_o) \exp[-\int_{t_o}^{t} a_1(u)\, du] + \qquad (2.3.4)$$
$$+ [\dot{y}(t_o) + a_1(t_o)\, y(t_o)] \int_{t_o}^{t} \exp[-\int_{v}^{t} a_1(u)\, du]\, dv$$

For the special initial conditions (2.2.27):

$$y(t_o) = y(\tau) = 0 \quad \text{and} \quad \dot{y}(t_o) = \dot{y}(\tau) = 1$$

the general solution (2.3.4) becomes the impulse response:

$$h(t,\tau) = \int_{\tau}^{t} \exp[-\int_{v}^{t} a_1(u)\, du]\, dv$$

2. Consider Euler's differential equation:

$$\ddot{y} + \frac{a_{1c}}{t+a}\dot{y} + \frac{a_{oc}}{(t+a)^2} y = 0 \qquad (2.3.5)$$

where a_{1c} and a_{oc} are constants. The following substitution:

$$y(t) = \eta(\xi) \quad , \quad \xi = \ln(t+a) \qquad (2.3.6)$$

reduces (2.3.5) to a differential equation with constant coefficients.

For the following specific numerical example:

$$\ddot{y} + \frac{4}{t+1}\dot{y} + \frac{2}{(t+1)^2} y = 0 \qquad (2.3.7a)$$

the substitution $\xi = \ln(t+1)$ yields the general solution:

$$y(t) = y(t_o)\ [1 + \frac{t-t_o}{t+1}]\ \frac{t_o+1}{t+1}$$
$$+ \dot{y}(t_o)\ [\frac{t_o+1}{t+1}]^2\ (t - t_o) \tag{2.3.7}$$

from which the impulse response is obtained as:

$$h(t,\tau) = [\frac{\tau+1}{t+1}]^2\ (t - \tau) \tag{2.3.8}$$

For time-invariant linear differential equations, the general solution and consequently $h(t,\tau)$ can conveniently be evaluated by using the Laplace transformation. Time-invariance implies the following property:

> $y(t)$, subject to certain initial conditions at $t = 0$, is equal to $y(t - t_o)$ for the same initial conditions at $t = t_o$.

Consider, for example,

$$\ddot{y} + a_{1c}\,\dot{y} + a_{oc}\,y = 0 \tag{2.3.9}$$

subject to

$$y(0) = y_o \quad \text{and} \quad \dot{y}(0) = \dot{y}_o$$

The solution of (2.3.9) can be obtained from:

$$y(t) = L^{-1}\left\{\frac{s+a_{1c}}{s^2+a_{1c}s+a_{oc}}\right\} y_o + L^{-1}\left\{\frac{1}{s^2+a_{1c}s+a_{oc}}\right\} \dot{y}_o \tag{2.3.10}$$

The solution of (2.3.9) for the delayed initial conditions $y(t_o) = y_o$ and $\dot{y}(t_o) = \dot{y}_o$ is simply $y(t - t_o)$ where $y(t)$ is defined in (2.3.10). This implies for the impulse response:

$$h(t,\tau) = h(t-\tau) \ , \quad \text{where} \quad h(t) = L^{-1} \left[\frac{1}{s^2+a_{1c}s+a_{oc}}\right]$$

Since the solution of a time-invariant differential equation can easily be determined it would seem logical to search for a transformation which reduces a time-varying system to a system with constant coefficients.

The substitution (2.3.7) for Euler's equation can be taken as one example of this kind of transformation. Muraview [6] proposes a more general method for a class of second order systems which will be dealt with in the remainder of this section.

Suppose the following differential equation

$$\ddot{y} + a_1(t)\,\dot{y} + a_o(t)\,y = 0 \qquad (2.3.11)$$

is defined for $t \geq T_o$ and is to be solved subject to the initial conditions $y(t_o)$ and $\dot{y}(t_o)$. The main idea of this method is to introduce the following transformation:

$$y(t) = \eta(\xi) \quad \text{and} \quad \xi = \int_{T_o}^{t} \frac{du}{\alpha(u)} \qquad (2.3.12)$$

In order to have a one-to-one relationship between ξ and t it will be assumed that $\alpha(u) > 0$. Differentiating (2.3.11) and denoting $\frac{d\eta}{d\xi} = \eta'$ gives:

$$\dot{y}(t) = \frac{1}{\alpha(t)}\,\eta'(\xi) \;, \quad \ddot{y}(t) = \frac{1}{\alpha^2(t)}\,\eta'' - \eta'\,\frac{\dot{\alpha}(t)}{\alpha^2(t)} \tag{2.3.13}$$

If (2.3.13) is substituted into (2.3.11), one gets:

$$\eta'' + \eta'\,[a_1(t)\,\alpha(t) - \dot{\alpha}(t)] + \eta[a_o(t)\cdot\alpha^2(t)] = 0 \tag{2.3.14}$$

In order to reduce (2.3.14) to the following time-invariant system:

$$\eta'' + \eta'\,m_1 + \eta\,m_o = 0 \tag{2.3.15}$$

two constants m_1 and m_o have to be determined such that:

$$\begin{aligned} \dot{\alpha}(t) - a_1(t)\,\alpha(t) &= -\,m_1 \\ a_o(t)\alpha^2(t) &= \;\; m_o \end{aligned} \tag{2.3.16}$$

Then (2.3.14) can be solved easily and the original solution $y(t)$ is obtained from $\eta(\xi)$ by substituting (2.3.12).

The method will be illustrated by the following example:

$$\ddot{y}(t) + 2\,\frac{t+1}{t^2}\,\dot{y} - 3\,\frac{1}{t^4}\,y = 0 \qquad \text{for} \qquad t \geq 1$$

Substituting:

$$a_1(t) = 2\,\frac{t+1}{t^2} \qquad \text{and} \qquad a_o(t) = -\,\frac{3}{t^4}$$

into (2.3.16) gives:

$$m_o = -\,\frac{3}{t^4}\,\alpha^2(t) \quad , \quad m_1 = 2\,\frac{t+1}{t^2}\,\alpha(t) - \dot{\alpha}\,(t)$$

The constants m_o and m_1 can, for instance, be chosen as:

$$m_o = -3 \quad , \quad m_1 = 2 \qquad (2.3.17)$$

so that

$$\alpha(t) = t^2 \quad \text{and} \quad \xi = 1 - \frac{1}{t} \qquad (2.3.18)$$

For the numerical values in (2.3.17), equation (2.3.15) has the general solution:

$$4\,\eta(\xi) = (\eta_o - \eta_o')\, e^{-3(\xi-\xi_o)} + (3\eta_o + \eta_o')\, e^{\xi-\xi_o}$$

which, observing (2.3.12) and (2.3.18), can be rewritten in terms of $y(t)$ and t as:

$$4\,y(t) = [y(t_o) - t_o^2\,\dot{y}(t_o)]\,\exp[-3\,\frac{t-t_o}{t\cdot t_o}] +$$
$$+ [3\,y(t_o) + t_o^2\,\dot{y}(t_o)]\,\exp[\frac{t-t_o}{t\cdot t_o}]$$

According to definition (2.2.27) the appropriate impulse response is:

$$4\,h(t,\tau) = \tau^2\,[\exp(\frac{t-\tau}{t\cdot\tau}) - \exp(-3\,\frac{t-\tau}{t\cdot\tau})]$$

As a concluding remark, the work by Kamke [7]+ should be mentioned which contains an extensive list of differential equations. This reference might be helpful in finding a closed form solution for a particular differential equation.

+see also the work by Murphy [7a]

CHAPTER III

APPROXIMATIONS

3.1 Introduction

In Chapter II some classes of linear systems were discussed which could be solved in closed form. Unfortunately, these systems represent exceptional cases and should not conceal the fact that arbitrary linear systems of higher than first order cannot be solved explicitly. Therefore $h(t,\tau)$ and consequently the convolution integral (1.2.8) cannot be obtained in closed form so that approximations have to be introduced.

For this purpose,(2.2.4) will be transformed into an equivalent integral equation. This will be discussed in section 3.2. It will be shown that the sequence of successive approximations converges for the assumptions made in Chapter II. In section 3.3, a bound on the absolute value of the impulse response will be derived. Finally, in section 3.4, several examples will illustrate the methods in section 3.2 and 3.3.

3.2 Linear Differential Equations and Linear Integral Equations

A linear differential equation, subject to certain initial conditions, can as well be represented by a linear Volterra integral equation of the second kind. It should be pointed out that representing the system by an integral equation does not imply a closed form

solution, and does not even imply an easier way to obtain the solution. However, as far as approximations are concerned, an integral equation offers a much wider range of methods than the original differential equation.

In order to illustrate the transformation from a differential equation into an integral equations in some detail, the following third order system will be used throughout this and the next section. Consider:

$$\dddot{y} + a_2(t)\,\ddot{y} + a_1(t)\,\dot{y} + a_o(t)\,y = 0 \tag{3.2.1}$$

defined for $t \geq t_o$ and subject to the initial conditions:

$$y(t_o) = y_o \quad , \quad \dot{y}(t_o) = \dot{y}_o \quad , \quad \ddot{y}(t_o) = \ddot{y}_o \tag{3.2.2}$$

In order to transform (3.2.1) into an integral equation, the following differential equation is introduced:

$$\dddot{y} + c_2(t)\,\ddot{y} + c_1(t)\,\dot{y} + c_o(t)\,y = 0 \tag{3.2.3}$$

where for the time being it will be assumed that the coefficients $c_o(t)$, $c_1(t)$, and $c_2(t)$ are "suitably" chosen and that (3.2.3) has a closed form solution.

Adding (3.2.3) to both sides of (3.2.1) and transposing the terms with $a_i(t)$ yields:

$$\dddot{y} + c_2(t)\, \ddot{y} + c_1(t)\, \dot{y} + c_o(t)\, y = \tag{3.2.4}$$

$$= \ddot{y}\, [c_2(t) - a_2(t)] + \dot{y}\, [c_1(t) - a_1(t)] + y\, [c_o(t) - a_o(t)]$$

By defining:

$$c_k(t) - a_k(t) = m_k(t) \;, \quad \text{for} \quad k = 0,\ 1,\ 2 \;, \tag{3.2.5}$$

equation (3.2.4) can be written as:

$$\dddot{y} + c_2(t)\, \ddot{y} + c_1(t)\, \dot{y} + c_o(t)\, y = \tag{3.2.6}$$

$$m_2(t)\, \ddot{y} + m_1(t)\, \dot{y} + m_o(t)\, y$$

Since the LHS of (3.2.6) was assumed to have a closed form solution, equation (3.2.6) can be expressed by means of convolution as:

$$y(t) = y_{LHS}(t) + \int_{t_o}^{t} g(t,u)\, [\ddot{y}(u)\, m_2(u) + \dot{y}(u)\, m_1(u) + \tag{3.2.7}$$

$$+ y(u)\, m_o(u)]\, du$$

where $y_{LHS}(t)$ is the solution of (3.2.3) subject to the initial conditions:

$$\frac{d^k}{dt^k} [y_{LHS}(t)]\bigg|_{t=t_o} = \frac{d^k}{dt^k} [y(t)]\bigg|_{t=t_o} \qquad \text{for} \qquad k = 0,\ 1,\ 2$$

and $g(t,u)$ is the impulse response of (3.2.3) satisfying:

$$\frac{\partial^3 g(t,u)}{\partial t^3} + c_2(t)\,\frac{\partial^2 g(t,u)}{\partial t^2} + c_1(t)\,\frac{\partial g(t,u)}{\partial t} + c_o(t)\,g(t,u) = 0 \tag{3.2.9}$$

for:

$$g(u,u) = 0 \quad , \quad \left.\frac{\partial g(t,u)}{\partial t}\right|_{t=u} = 0 \ , \quad \left.\frac{\partial^2 g(t,u)}{t^2}\right|_{t=u} = 1$$

From (2.2.28) it follows in addition:

$$\left.\frac{\partial g(t,u)}{\partial u}\right|_{t=u} = 0$$

Integrating (3.2.7) by parts gives:

$$\begin{aligned} y(t) = {} & y_{LHS}(t) + \int_{t_o}^{t} y(u)\,m_o(u)\,g(t,u)\,du \; + \\ & + y(u)\cdot m_1(u)\cdot g(t,u)\,\Big|_{t_o}^{t} - \int_{t_o}^{t} y(u)\,\frac{\partial}{\partial u}[m_1(u)\;g(t,u)]\,du \\ & + \dot{y}(u)\cdot m_2(u)\cdot g(t,u)\,\Big|_{t_o}^{t} - y(u)\cdot\frac{\partial}{\partial u}\,[m_2(u)\;g(t,u)]\,\Big|_{t_o}^{t} \\ & + \int_{t_o}^{t} y(u)\,\frac{\partial^2}{\partial u^2}\,[m_2(u)\;g(t,u)]\,du \end{aligned} \tag{3.2.10}$$

or observing (3.2.9)

$$y(t) = f(t) + \int_{t_o}^{t} k(t,u)\,y(u)\,du \tag{3.2.11}$$

where

$$f(t) = y_{LHS}(t) + y(t_o) \frac{\partial}{\partial u} [m_2(u)\ g(t,u)] \Big|_{u=t_o} \tag{3.2.12}$$

$$- \dot{y}(t_o)\ m_2(t_o)\ g(t,t_o) - y(t_o)\ m_1(t_o)\ g(t,t_o)$$

and

$$k(t,u) = \frac{\partial^2}{\partial u^2} [m_2(u)\ g(t,u)] - \frac{\partial}{\partial u} [m_1(u)\ g(t,u)] + m_o(u) \cdot g(t,u) \tag{3.2.13}$$

According to definition (2.2.27) the general solution (3.2.11) becomes the impulse response if the following initial conditions are substituted at $t_o = \tau$:

$$y(\tau) = 0\ ,\ \dot{y}(\tau) = 0\ ,\ \ddot{y}(\tau) = 1 \tag{3.2.14}$$

which gives:

$$h(t,\tau) = g(t,\tau) + \int_{\tau}^{t} k(t,u)\ h(u,\tau)\ du \tag{3.2.15}$$

Equation (3.2.15) can be interpreted in the following way:

> The impulse response of (3.2.1) is equal to the impulse response $g(t,\tau)$ of the LHS of (3.2.6) plus a "correction" term $\int_{\tau}^{t} \cdots du$, due to the RHS of (3.2.6).

The integral equations (3.2.11) and (3.2.15) are equivalent to the differential equation (3.2.1) for the initial conditions (3.2.2)

and (3.2.14) respectively. As mentioned earlier, the integral representation does not imply an easier way to a possible explicit solution. However, it is a prerequisite to prove convergence of successive iterations if the solution of (3.2.1) is to be approximated in this manner.

The standard approximation method, sometimes called "Picard's method of successive approximations", proceeds as follows:

Suppose $y(t)$ is to be approximated on the interval $[t_o = 0\ , T]$. For this purpose, a sequence of partial sums $\{y_n = \sum_{o}^{n} k\, s_k\}$ will be generated from (3.2.11) as:

$$y_o(t) = f(t) = s_o(t) \ , \tag{3.2.16}$$

$$y_1(t) = f(t) + \int_o^t k(t,u)\ y_o(u)\ du = s_o(t) + s_1(t) \ ,$$

or in general:

$$\begin{aligned} y_{n+1}(t) &= f(t) + \int_o^t k(t,u)\ y_n(u)\ du \\ &= s_o(t) + s_1(t) + \cdots + s_n(t) + s_{n+1}(t) \end{aligned} \tag{3.2.17}$$

The difference of two successive approximations can be expressed as:

$$\begin{aligned} s_{n+1} = y_{n+1} - y_n &= \int_o^t k(t,u)\ [y_n(u) - y_{n-1}(u)]\ du \\ &= \int_o^t k(t,u)\ s_n(u)\ du \end{aligned} \tag{3.2.18}$$

In order that the terms y_n form better approximations of y as n increases, it must be proved that:

$$y(t) = \lim_{n\to\infty} \{y_n(t)\} \quad \text{on} \quad [0,T] \tag{3.2.19}$$

Then $y(t)$ can be written as:

$$y(t) = s_o(t) + s_1(t) + \cdots = \sum_o^\infty k\, s_k(t) \ . \tag{3.2.20}$$

where s_o and s_k $(k = 1,2,\cdots)$ are defined in (3.2.16) and (3.2.18).

Chu and Diaz [8] use an elegant method to prove the convergence of (3.2.20) which will be reproduced here in a slightly different way.

Multiplying both sides of (3.2.18) by $e^{-\lambda t}$ for some $\lambda > 0$ gives:

$$s_{n+1}\, e^{-\lambda t} = \int_o^t k(t,u)\, e^{-\lambda t}\, e^{\lambda u}\, e^{-\lambda u}\, s_n(u)\, du \tag{3.2.21}$$

By defining:

$$y(t)\, e^{-\lambda t} = z(t) \ , \quad \text{and} \quad s_k(t)\, e^{-\lambda t} = z_k(t) \tag{3.2.22}$$

$$\text{for} \quad k = 0,1,2,\cdots$$

equation (3.2.21) becomes:

$$z_{n+1} = \int_o^t k(t,u)\, e^{-\lambda t}\, e^{+\lambda u}\, z_n(u)\, du \tag{3.2.23}$$

The absolute value of (3.2.23)

$$|z_{n+1}| \leq \int_o^t |k(t,u)| \, e^{-\lambda(t-u)} \, |z_n(u)| \, du \qquad (3.2.24)$$

satisfies the following inequality:

$$|z_{n+1}| \leq \sup_{[0,T]} \{|z_n(t)|\} \int_o^t |k(t,u)| \, e^{-\lambda(t-u)} \, du \qquad (3.2.25)$$

If there exists a positive λ such that:

$$\sup_{[0,T]} \{\int_o^t |k(t,u)| \, e^{-\lambda(t-u)} \, du\} \leq \alpha < 1 \qquad (3.2.26)$$

which can be satisfied, for instance, if $k(t,u)$ is continuous on $0 \leq u \leq t \leq T$, then (3.2.23) can be bounded from above by:

$$\sup_{[0,T]} \{|z_{n+1}|\} \leq \alpha \sup_{[0,T]} \{|z_n|\} \quad \text{for} \quad n = 0,1,2\cdots \qquad (3.2.27)$$

Equation (3.2.27) implies that the $\sup\text{-norm}_{[0,T]}$ of the series:

$$z(t) = z_o(t) + z_1(t) + z_2(t) + \cdots \qquad (3.2.28)$$

can be simplified as follows:

$$\sup_{[0,T]} \{|z(t)|\} \leq \sup_{[0,T]} \{|z_o(t)|\} + \alpha \sup_{[0,T]}\{|z_o(t)|\} + \alpha^2 \sup_{[0,T]}\{|z_o(t)|\} +$$

$$\cdots = (1 + \alpha + \alpha^2 + \cdots) \sup_{[0,T]} \{|z_o(t)|\} \qquad (3.2.29)$$

$$= \frac{1}{1-\alpha} \sup_{[0,T]} \{z_o(t)\}$$

The difference between z and the sum of the first $N-1$ terms in the series (3.2.28) is:

$$z(t) - \sum_{o}^{N-1} k\, z_k = \sum_{N}^{\infty} k\, z_k(t)$$

which can be bounded from above by:

$$\sup_{[0,T]} \{|z(t) - \sum_{o}^{N-1} k\; z_k|\} \leq \frac{\alpha^N}{1 - \alpha} \sup_{[0,T]} \{|z_o|\} \qquad (3.2.30)$$

Since $\alpha < 1$, the coefficient α^N can be made arbitrarily small, provided N is large enough.

Let α^N be, for example:

$$\alpha^N < \varepsilon \frac{1 - \alpha}{M} e^{-\lambda T} \qquad (3.2.31)$$

where

$$M = \sup_{[0,T]} \{|z_o(t)|\} = \sup_{[0,T]} \{|f(t)|\, e^{-\lambda t}\} \qquad (3.2.32)$$

Recalling definition (3.2.22), equation (3.2.30) can be rewritten in terms of $y(t)$ as:

$$\sup_{[0,T]} \{|y(t) - \sum_{o}^{N-1} k\; s_k| e^{-\lambda t}\} \leq \frac{\alpha^N}{1 - \alpha} M \qquad (3.2.33)$$

Making use of (3.2.31), the last expression implies:

$$|y(t) - \sum_{o}^{N-1} k\; s_k| < \varepsilon \quad \text{on} \quad [0,T] . \qquad (3.2.34)$$

which completes the proof.

Equation (3.2.34) can be interpreted as follows: The series (3.2.34) converges uniformly on every finite interval $[0,T]$, provided (3.2.26) is satisfied. This implies that $y(t)$ can be approximated by a finite number of terms $s_k(t)$ such that the error $\left|y(t) - \sum_{o}^{N-1} k\ s_k(t)\right|$ does not exceed a given ε. Therefore, from the convergence point of view, the iterative approximations, defined in (3.2.16) and (3.2.18), are completely justified.

However, if the terms $s_k(t)$ are to be determined from (3.2.16) and (3.2.18), one will find in many cases that the evaluation generates too voluminous expressions or integrals which cannot be computed in closed form. Due to this difficulty, the result of the last proof can in general not be exploited to obtain an arbitrarily accurate approximation.

3.3 Integral Inequality

A less demanding, but for later purposes still acceptable, approximation is a bound on $|y(t)|$. Taking the modulus of (3.2.11) yields the following inequality:

$$|y(t)| \leq |f(t)| + \int_o^t |k(t,u)|\ |y(u)|\ du \tag{3.3.1}$$

While (3.2.11) defines a unique function, the inequality (3.3.1) is valid for a set of functions, one of which is $|y(t)|$.

If $f(t)$ and $k(t,u)$ satisfy the following conditions:

$$f(t) \geq 0 \quad \text{and} \quad k(t,u) \geq 0 \quad \text{for} \quad 0 \leq u \leq t \leq T \qquad (3.3.2)$$

the equality sign holds in (3.3.1) and the bound on $|y(t)|$ is identical to $y(t)$. This is of course the ideal case. Therefore it is desirable to transform the given differential equation into an integral equation which has the properties specified in (3.3.2). One possible way of achieving this may be accomplished by the choice of the coefficients $c_o(t)$, $c_1(t)$, and $c_2(t)$ in (3.2.3). Now, one aspect of a suitable choice can be specified as follows: "Choose the coefficients so that the resulting integral equation is not changed 'too much' by taking the modulus".

The fundamental idea to determine a bound on (3.3.1) can be formulated in Theorem I:

Theorem I:

Let (3.3.1) be given, that is:

$$|y(t)| \leq |f(t)| + \int_o^t |k(t,u)|\,|y(u)|\, du \qquad (3.3.3)$$

Then, for every $Y(t)$ which is defined by:

$$Y(t) = F(t) + \int_o^t K(t,u)\, Y(u)\, du \qquad (3.3.4)$$

where:

$$\left.\begin{aligned} 0 &\leq |f(t)| \leq F(t) \\ 0 &\leq |k(t,u)| \leq K(t,u) \end{aligned}\right\} \quad \text{for} \quad 0 \leq u \leq t \leq T$$

the following property is true:

$$|y(t)| \leq Y(t) \quad \text{for} \quad 0 \leq t \leq T \tag{3.3.5}$$

In order to prove this seemingly self-evident fact, some basic properties of linear integral equations are reviewed.

Suppose the following integral equation is defined on the interval $[0,T]$:

$$z(t) = s(t) + \int_0^t q(t,u)\, z(u)\, du \tag{3.3.6}$$

(a) If

$$s(t) \equiv 0 \quad \text{on} \quad [0,T]$$

then

$$z(t) \equiv 0 \quad \text{on} \quad [0,T] .$$

This is proved by Tricomi [9], for example.

(b) Let in (3.3.6):

$$s(t) \geq 0 \quad \text{and} \quad q(t,u) \geq 0 \quad \text{on} \quad [0,T]$$

Then

$$|z(t)| - z(t) = |s(t)| - s(t) + \int_0^t q(t,u)\, [|z(u)| - z(u)]\, du$$

Since

$$|s(t)| - s(t) \equiv 0$$

it follows from (a) that:

$$z(t) \geq 0 \quad \text{on} \quad [0,T] \ .$$

(c) Assume for (3.3.6):

$$q(t,u) \geq 0 \quad \text{and} \quad s(t) \quad \text{arbitrary.}$$

Then every $z_i(t)$ which satisfies:

$$z_i(t) \leq s(t) + \int_o^t q(t,u)\ z_i(u)\ du \tag{3.3.7}$$

is bounded from above in the following way:

$$z_i(t) \leq z(t)$$

where $z(t)$ is the solution of:

$$z(t) = s(t) + \int_o^t q(t,u)\ z(u)\ du \tag{3.3.8}$$

This can be proved by introducing a function $p(t) \geq 0$ such that (3.3.7) becomes an integral equality:

$$z_i(t) = s(t) - p(t) + \int_o^t q(t,u)\ z_i(u)\ du$$

The difference $z(t) - z_i(t)$ satisfies:

$$z(t) - z_i(t) = p(t) + \int_o^t q(t,u)\ [z(u) - z_i(u)]\ du$$

so that it follows from (b):

$$z(t) - z_i(t) \geq 0 \quad \text{or} \quad z(t) \geq z_i(t)$$

Utilizing (b) and (c), Theorem I is now readily proved. Because of (c) every $|y(t)|$ which satisfies (3.3.3) is bounded from above by:

$$|y(t)| \leq y_e(t)$$

where $y_e(t)$ is the solution of the following integral equality:

$$y_e(t) = |f(t)| + \int_o^t |k(t,u)|\ y_e(u)\ du$$

Therefore it is sufficient to show that:

$$Y(t) - y_e(t) \geq 0$$

Subtracting $y_e(t)$ from (3.3.4) gives:

$$\begin{aligned} Y(t) - y_e(t) &= F(t) - |f(t)| + \int_o^t K(t,u)\ Y(u)\ du - \int_o^t |k(t,u)|\ y_e(u)\ du \\ &= F(t) - |f(t)| + \int_o^t [K(t,u) - |k(t,u)|]\ y_e(u)\ du \\ &\qquad + \int_o^t K(t,u)\ [Y(u) - y_e(u)]\ du \end{aligned}$$

Since $y_e(t) \geq 0$ and $K(t,u) - |k(t,u)| \geq 0$ it follows from (b):

$$Y(t) - y_e(t) \geq 0$$

which completes the proof of Theorem I.

In order to apply Theorem I, $Y(t)$ has to be determined in closed form. The simplest form of $K(t,u)$ which allows a closed form solution is:

$$K(t,u) \equiv K(u) \qquad \text{for} \qquad t \in [0,T]$$

so that (3.3.4) can be simplified to:

$$Y(t) = F(t) + \int_o^t K(u)\, Y(u)\, du \tag{3.3.9}$$

By differentiating once and integrating, the solution of (3.3.9) becomes:

$$\begin{aligned} Y(t) &= Y(0)\, e^{\int_o^t K(u)\, du} + \int_o^t \dot{F}(u)\, e^{\int_u^t K(v)\, dv}\, du \\ &= F(t) + \int_o^t F(u)\, K(u)\, e^{\int_u^t K(v)\, dv}\, du \end{aligned} \tag{3.3.10}$$

Equation (3.3.10), together with property (c) represents a generalization of various forms of the Bellman-Gronwall lemma which can be found in Hsu and Meier [10], Sansone and Conti [11], and Halany [12].

Halany, for example, shows that

if:

$$z_i \leq s(t) + \int_o^t q(u)\, z_i(u)\, du \tag{3.3.11}$$

with

$$q(u) \geq 0$$

then

$$z_i(t) \leq s(t) + \int_0^t q(u)\, s(u)\, e^{\int_u^t q(v)\, dv}\, du \,. \quad (3.3.12)$$

Inequality (3.3.12) can easily be verified by the methods developed here.

From property (c) it follows:

$$z_i(t) \leq z(t) \quad (3.3.13)$$

where

$$z(t) = s(t) + \int_0^t q(u)\, z(u)\, du \,. \quad (3.3.14)$$

Equation (3.3.14) can be solved in the same manner as (3.3.9) in order to get:

$$z(t) = s(t) + \int_0^t q(u)\, s(u)\, e^{\int_u^t q(v)\, dv}\, du \quad (3.3.15)$$

Combining (3.3.13) and (3.3.15) gives (3.3.12) which completes the proof.

Another class of integral equations which has a closed form solution has the following kernel:

$$K(t,u) = K(t - u)$$

The appropriate integral equation (3.3.4) can conveniently be solved by using Laplace transform (see Doetsch [13]). Taking the Laplace transformation of:

$$Y(t) = F(t) + \int_0^t K(t-u)\, Y(u)\, du$$

gives:

$$Y(s) = F(s) + K(s)\, Y(s) ,$$

or

$$Y(s) = \frac{F(s)}{1 - K(s)} ,$$

so that

$$Y(t) = L^{-1} \{ \frac{F(s)}{1 - K(s)} \} .$$

Two special cases which will be of importance for the next section will be now discussed in more detail. Consider:

1. $K(t - u) = \sum_{i=0}^{n-1} k_i (t-u)^i$

The appropriate integral equation is:

$$Y(t) = F(t) + \int_0^t \sum_{i=0}^{n-1} k_i (t-u)^i\, Y(u)\, du \tag{3.3.16}$$

Differentiating (3.3.16) n times gives an n-th order differential equation with constant coefficients which can be solved in the customary manner.

2. $K(t-u) = k_1\, e^{\lambda_1(t-u)} + k_2\, e^{\lambda_2(t-u)}$

The corresponding integral equation:

$$Y(t) = F(t) + \int_o^t [k_1\, e^{\lambda_1(t-u)} + k_2\, e^{\lambda_2(t-u)}]\, Y(u)\, du$$

can be rewritten as:

$$Y(t)\, e^{-\lambda_1 t} = F(t)\, e^{-\lambda_1 t} + \int_o^t [k_1 + k_2\, e^{(\lambda_2-\lambda_1)(t-u)}] Y(u)\, e^{-\lambda_1 u}\, du \tag{3.3.17}$$

Defining

$$Y(t)\, e^{-\lambda_1 t} = Z(t)\ ,\ F(t)\, e^{-\lambda_1 t} = F_1(t)\ , \qquad \lambda_2 - \lambda_1 = \lambda \tag{3.3.18}$$

transforms (3.3.17) into

$$Z(t) = F_1(t) + \int_o^t [k_1 + k_2\, e^{\lambda(t-u)}]\, Z(u)\, du \tag{3.3.19}$$

Differentiating (3.3.19) twice gives:

$$\dot{Z} = \dot{F}_1 + (k_1 + k_2)\, Z + \int_o^t k_2\, \lambda\, e^{\lambda(t-u)}\, Z(u)\, du$$

$$\ddot{Z} = \ddot{F}_1 + (k_1 + k_2)\, \dot{Z} + k_2\lambda\, Z + \int_o^t k_2\, \lambda^2\, e^{\lambda(t-u)} Z(u)\, du$$

or

$$\ddot{Z} - \lambda\dot{Z} = \ddot{F}_1 - \lambda\, \dot{F}_1 + \dot{Z}(k_1 + k_2) - Z\, (k_1\lambda)$$

or

$$\ddot{Z} - \dot{Z}\, (\lambda + k_1 + k_2) + k_1\, \lambda\, Z = \ddot{F}_1 - \lambda\, \dot{F}_1 \tag{3.3.20}$$

Equation (3.3.20) is a differential equation with constant coefficients. Therefore $Z(t)$ can be solved in closed form. Substituting $Z(t)$ into (3.3.18) gives $Y(t)$.

Before the methods, discussed in the last two sections, are applied to certain examples, the basic procedure will be reviewed briefly.

The differential equation (3.2.1) is given for $y(t)$.

Equation (3.2.1) is converted into a suitable integral equation (3.2.11) for $y(t)$.

Taking the modulus transforms (3.2.11) into the integral inequality (3.3.1) for $|y(t)|$.

Inequality (3.3.1) is compared with the integral equation (3.3.4) which satisfies the conditions of Theorem I.

The comparison gives the desired result: $|y(t)| \leq Y(t)$.

3.4 Examples

In this section, several examples will illustrate the methods, which were described in general in section 3.2 and 3.3. The efficiency of these methods depends to a large extent on equation (3.2.3) which has to satisfy two requirements. Firstly, it must have a closed form solution, secondly, the corresponding integral equation must yield an acceptable bound on the appropriate solution. Unfortunately, there is no general rule according to which (3.2.3) could be determined so that the previous requirements are satisfied and the best possible solution

is obtained for every particular problem. If an attempt is made to choose suitable time-varying coefficients $c_k(t)$ $(k = 0,1,2)$ for equation (3.2.3), one will find that this approach exceeds the scope of this thesis. Consequently, the discussion in this section will be limited to a stationary LHS approach.

To start with, a numerical example will be used to reproduce the steps which were discussed in general in section 3.2 and 3.3.

Suppose the following differential equation is given:

$$\ddot{y} + (2 - \frac{1}{3}\cos t)\,\dot{y} + (1 - \frac{2}{3}\cos t)\,y = 0 \qquad (3.4.1)$$

subject to the initial conditions:

$$y(t_o) = y_o\,, \quad \text{and} \quad \dot{y}(t_o) = \dot{y}_o$$

According to (3.2.4), two constants c_o and c_1 have to be introduced so that (3.4.1) becomes:

$$\ddot{y} + c_1\dot{y} + c_o y = \dot{y}[c_1 - 2 + \frac{1}{3}\cos t] + y[c_o - 1 + \frac{2}{3}\cos t] \qquad (3.4.2)$$

At this point, it cannot be determined yet which values have to be substituted for c_1 and c_o in order to get a good approximation. Hence, for the time being, two values will be taken which give a simple expression for (3.4.2).

Let:

$$c_1 = 2 \quad \text{and} \quad c_o = 1$$

Then (3.2.6) can be written as:

$$\ddot{y} + 2\,\dot{y} + y = \dot{y}\,\frac{1}{3}\cos t + y\,\frac{2}{3}\cos t$$

The solution of the LHS is:

$$y_{LHS} = y_o\, e^{-(t-t_o)} + (y_o + \dot{y}_o)(t - t_o)\, e^{-(t-t_o)}$$

Thus:

$$g(t,u) = g(t-u) = (t-u)\, e^{-(t-u)}$$

Substituting y_{LHS} and $g(t,u)$ into (3.2.10) gives:

$$y(t) = e^{-(t-t_o)}\, f_1(t) + \int_{t_o}^{t} e^{-(t-u)}\, k_1(t,u)\, y(u)\, du \qquad (3.4.3)$$

where

$$f_1(t)\, e^{-(t-t_o)} = y_o\, e^{-(t-t_o)}\,[1 + (t-t_o)(1 - \frac{1}{3}\cos t_o)]$$

$$+ \dot{y}_o\,[(t-t_o)\, e^{-(t-t_o)}]$$

$$k_1(t,u)\, e^{-(t-u)} = e^{-(t-u)}\,[\frac{1}{3}(t-u)(\cos u + \sin u) + \frac{1}{3}\cos u]$$

By introducing the special initial conditions:

$$t_o = \tau \quad , \quad y(t_o = \tau) = 0 \quad , \quad \dot{y}(t_o = \tau) = 1$$

an integral equation is obtained for the impulse response of (3.4.1) as:

$$h(t,\tau) = (t-\tau)\, e^{-(t-\tau)} + \int_{\tau}^{t} e^{-(t-u)}\, k_1(t,u)\, h(u,\tau)\, du \quad (3.4.4)$$

Equation (3.4.3) can be simplified by the following definition:

$$z(t) = y(t)\, e^{t-t_o} \quad (3.4.5)$$

so that (3.4.3) becomes:

$$z(t) = f_1(t) + \int_{t_o}^{t} k_1(t,u)\, z(u)\, du$$

Similarly for (3.4.4) it is convenient to define:

$$l(t,\tau) = h(t,\tau)\, e^{(t-\tau)} \quad (3.4.6)$$

which results in:

$$l(t,\tau) = (t-\tau) + \int_{\tau}^{t} k_1(t,u)\, l(u,\tau)\, du \quad (3.4.7)$$

If an attempt is made to solve (3.4.7) by successive iterations as defined in (3.2.16), the terms in the series (3.2.20):

$$l(t,\tau) = l_o(t,\tau) + l_1(t,\tau) + l_2(t,\tau) + \cdots$$

are evaluated as follows:

$$l_o(t,\tau) = (t-\tau)$$

$$l_1(t,\tau) = \int_\tau^t k_1(t,u)\ [u-\tau]\ du$$

$$= \frac{1}{3}\ (\tau-t)\ [\cos t + \cos \tau\ + \sin \tau]$$

$$+ \frac{2}{3}\ [\sin t - \sin \tau] + \frac{1}{3}\ [\cos \tau - \cos t]$$

$$l_2(t,\tau) = \int_\tau^t k_1(t,u)\ l_1(u,\tau)\ du$$

Although $l_2(t,\tau)$ and all subsequent terms can be evaluated in closed form, the size of the integral makes it advisable to abandon this approach and to determine a bound on the modulus of $l(t,\tau)$.

Taking the absolute value of (3.4.7) yields the following inequality:

$$\left|l(t,\tau)\right| \le \left|t-\tau\right| + \frac{1}{3}\int_\tau^t \left|[(t-u)(\cos u + \sin u) + \cos u]\right|\left|l(u,\tau)\right|\ du \tag{3.4.8}$$

For the domain $0 \le \tau \le t < \infty$, which is of interest for the convolution integral (1.2.8), equation (3.4.8) can be rewritten as:

$$|l(t,\tau)| \le [t-\tau] + \frac{1}{3}\int_\tau^t [(t-u)\cdot|\cos u + \sin u| + |\cos u|]\ |l(u,\tau)|\ du$$

According to Theorem I, $|l(t,\tau)|$ is bounded by:

$$|l(t,\tau)| \le L(t,\tau)$$

where

$$L(t,\tau) = (t-\tau) + \frac{1}{3}\int_\tau^t [(t-u)\ \sqrt{2} + 1]\ L(u,\tau)\ du \tag{3.4.9}$$

Differentiating (3.4.9) twice with respect to t gives:

$$\ddot{L}(t,\tau) - \frac{1}{3}\dot{L}(t,\tau) - \frac{\sqrt{2}}{3}L(t,\tau) = 0 \; , \qquad \begin{array}{l} L(\tau,\tau) = 0 \\ \dot{L}(t,\tau)\Big|_{t=\tau} = 1 \end{array}$$

which can be solved as:

$$\frac{8.5}{6}L(t,\tau) = e^{\frac{5.25}{6}(t-\tau)} - e^{-\frac{3.25}{6}(t-\tau)}$$

Recalling (3.4.6):

$$|1(t,\tau)| = |h(t,\tau)|\, e^{+(t-\tau)} \leq L(t,\tau)$$

yields the following bound on the impulse response:

$$|h(t,\tau)| \leq \frac{6}{8.5}\,[e^{-0.125(t-\tau)} - e^{-1.54(t-\tau)}] \qquad (3.4.10)$$

In a similar manner, a bound on the general solution can be obtained as:

$$|y(t)| \leq \frac{13.25|y_o| + 6|\dot{y}_o|}{8.5}\, e^{-0.125(t-t_o)} - \frac{4.75|y_o| + 6|\dot{y}_o|}{8.5}\, e^{-1.54(t-t_o)} \qquad (3.4.11)$$

Considering the accuracy of the last result one might suggest that different numerical values for c_o and c_1 in (3.4.2) could lead to a better bound on $|h(t,\tau)|$. While the last choice

gave a double pole at -1, the following decomposition, using an auxiliary parameter ω :

$$\ddot{y} + 2\,\dot{y} + (1 \pm \omega^2)\,y = \frac{1}{3}\,\dot{y}\cos t + [\pm\,\omega^2 + \frac{2}{3}\cos t]\,y \qquad (3.4.12)$$

generates two distinct poles for the LHS of (3.4.12) and might improve (3.4.10) for some $\omega \neq 0$. This has been tried without success and (3.4.10) and (3.4.11) appear to be the best possible approximations that can be obtained by this - stationary LHS - approach.

In the last example, the stationary LHS approach was used to determine the stability of a given system. Now the class of systems will be outlined which can be treated by this approach.

Suppose an acceptable result is a bound of the following form:

$$y(t) \leq C_o\, e^{-\rho t} \qquad (3.4.13)$$

where ρ and C_o are positive constants.

In order that (3.4.13) can be satisfied for a stationary LHS (3.2.3), it is necessary that (3.2.3) has poles s_i in the left half plane only.

Thus

$$\mathrm{Re}\{s_i\} \leq -\delta \quad , \quad \text{for} \quad \begin{array}{l} i = 1,2,3 \\ \delta > 0 \end{array}$$

The impulse response $g(t,u) = g(t-u)$, defined in (3.2.9), is a special solution of (3.2.3). Consequently:

$$\frac{\partial^k g(t-u)}{\partial u^k} \leq G_o \; e^{-\delta(t-u)} \quad , \quad \text{for} \quad \begin{array}{l} k = 0,1,2 \\ G_o > 0 \end{array}$$

If the kernel $k(t,u)$ in (3.2.13) is expanded in the following manner:

$$\begin{aligned} k(t,u) = \quad & g(t-u)\,[m_o(u) - \dot{m}_1(u) + \ddot{m}_2(u)] \\ & + \frac{\partial}{\partial u}\{g(t-u)\}[\quad - m_1(u) + 2\dot{m}_2(u)] \qquad (3.4.14) \\ & + \frac{\partial^2}{\partial u^2}\{g(t-u)\}[\qquad + m_2(u)] \end{aligned}$$

and if the terms $\dfrac{d^i m_j(u)}{du^i}$ are bounded by some constants in the following manner:

$$\begin{array}{lll} |m_o| \leq k_{00} \; , & & \\ |m_1| \leq k_{10} \; , & |\dot{m}_1| \leq k_{11} \; , & \\ |m_2| \leq k_{20} \; , & 2|\dot{m}_2| \leq k_{21} \; , & |\ddot{m}_2| \leq k_{22} \end{array} \qquad (3.4.15)$$

then $k(t,u)$ can be bounded by:

$$k(t,u) \leq \sum_i \sum_j k_{ij} \; G_o \; e^{-\delta(t-u)} = k_o \; e^{-\delta(t-u)} \qquad (3.4.16)$$

Using (3.4.16), inequality (3.3.1) can be written as:

$$|y(t)| \leq |f(t)| + \int_{t_o}^{t} k_o \; e^{-\delta(t-u)} \; |y(u)| \; du \qquad (3.4.17)$$

In order that $|y(t)|$ in (3.4.17) is bounded by a function of type (3.4.13), it is required that:

$$k_o = \sum_i \sum_j k_{ij} \; G_o \leq \delta \tag{3.4.18}$$

Naturally, this simple derivation results in a very conservative bound (3.4.18) for the time-varying parameters. However, (3.4.15) represents the essential prerequisite, to apply any stationary LHS approach. To state the result differently, if one of the terms in (3.4.15) is not bounded, this approach cannot be used.

In the remainder of this section, the stationary LHS approach will be applied to two further examples. In the first example, the following 3rd order differential equation of Popov type will be considered.

$$\dddot{y} + 6\,\ddot{y} + 12\,\dot{y} + a(t)\,y = 0 \tag{3.4.19}$$

Suppose, a range of $a(t)$ is to be determined such that the solution of (3.4.19) is stable.

For this purpose, (3.4.19) is first rewritten in the following form: (this is sometimes called pole-shifting)

$$\dddot{y} + 6\,\ddot{y} + 12\,\dot{y} + 8\,y = [8 - a(t)]\,y \tag{3.4.20}$$

The characteristic equation of the LHS of (3.4.20):

$$\lambda^3 + 6\,\lambda^2 + 12\,\lambda + 8 = 0$$

has the following solutions: $\lambda_1 = \lambda_2 = \lambda_3 = -2$.

Consequently, the general solution of the LHS of (3.4.20) is of the form:

$$y_{LHS} = A\,e^{-2t} + B\,t\,e^{-2t} + C\,t^2\,e^{-2t} \tag{3.4.21}$$

where A , B , and C include the initial conditions at $t_o = 0$. The appropriate impulse response is:

$$g(t-u) = \frac{(t-u)^2}{2}\,e^{-2(t-u)} \tag{3.4.22}$$

Utilizing (3.4.21) and (3.4.22) gives the following integral equation for (3.4.20):

$$y(t) = A\,e^{-2t} + B\,t\,e^{-2t} + C\,t^2\,e^{-2t} + \frac{1}{2}\int_o^t (t-u)^2\,[8 - a(u)]\,e^{-2(t-u)}\,y(u)\,du \tag{3.4.23}$$

By defining:

$$y(t)\,e^{2t} = z(t) \tag{3.4.24}$$

equation (3.4.23) can be simplified to:

$$z(t) = A + B\,t + C\,t^2 + \frac{1}{2}\int_o^t (t-u)^2\,[8-a(u)]\,z(u)\,du \tag{3.4.25}$$

Taking the modulus of (3.4.25) yields:

$$|z(t)| \leq |A| + |B|\,t + |C|\,t^2 + \frac{1}{2}\int_o^t (t-u)^2\,|[8-a(u)]|\,|z(u)|\,du \tag{3.4.26}$$

Now Theorem I can be applied to (3.4.26) which results in:

$$|z(t)| \leq Z(t)$$

where

$$Z(t) = |A| + |B|\ t + |C|\ t^2 + \frac{1}{2}\int_0^t (t-u)^2\ m_o\ Z(u)\ du \qquad (3.4.27)$$

with

$$|8 - a(u)| \leq m_o = \text{const.} \qquad (3.4.28)$$

If (3.4.27) is differentiated three times, one gets:

$$\dddot{Z}(t) - m_o\ Z(t) = 0 \qquad \begin{aligned} Z(0) &= |A| \\ \dot{Z}(0) &= |B| \\ \ddot{Z}(0) &= 2|C| \end{aligned} \qquad (3.4.29)$$

The characteristic equation of (3.4.29) is:

$$\lambda^3 - m_o = 0$$

which has the following solutions:

$$\lambda_1 = \sqrt[3]{m_o}\ , \quad \lambda_2 = \sqrt[3]{m_o}\ e^{-j2\pi/3}\ , \quad \lambda_3 = \sqrt[3]{m_o}\ e^{+j2\pi/3} \qquad (3.4.30)$$

Therefore $Z(t)$ has the following general solution:

$$Z(t) = Z_{10}\ e^{\lambda_1 t} + Z_{20}\ e^{\lambda_2 t} + Z_{30}\ e^{\lambda_3 t} \qquad (3.4.31)$$

where the constants Z_{io} can be determined from the initial conditions (3.4.29). Observing (3.4.24) gives:

$$|y(t)|\ e^{2t} = |z(t)| \leq Z(t)$$

so that (3.4.31) can be rewritten in terms of $y(t)$ as:

$$|y(t)| \leq Z_{10}\ e^{(\lambda_1-2)t} + Z_{20}\ e^{(\lambda_2-2)t} + Z_{30}\ e^{(\lambda_3-2)t}$$

If $y(t)$ is to be bounded by a decreasing exponential, it must be required that:

$$\lambda_1 < 2$$

or because of (3.4.28) and (3.4.30):

$$|8 - a(u)| \leq m_o = \lambda_1^3 < 8$$

Therefore an admissible range for $a(t)$ is:

$$0 < a(t) < 16 \qquad (3.4.32)$$

For comparison, an admissible range for $a(t)$ is determined from some customary stability criteria. The results are listed below:

Hurwitz sector:	$0 < a(t) = \text{const} < 72$
Popov criterion: (as given by Hsu and Meier [14])	$0 < a(t) < 12$
Bellman Gronwall lemma:	$0 < a(t) < 4.4$
This method:	$0 < a(t) < 16$

While in the last example the stationary LHS had a multiple root, the next example will illustrate how to use a LHS with distinct

roots.

Suppose the following differential equation is defined for $t \geq t_o$:

$$\ddot{y} + (3 - 6\,e^{-t})\,\dot{y} + (2 - 4\,e^{-t})\,y = 0 \qquad (3.4.33)$$

subject to the initial conditions:

$$y(t_o) = y_o \quad , \quad \dot{y}(t_o) = \dot{y}_o$$

For large t_o , (3.4.33) approaches the following equation:

$$\ddot{y} + 3\,\dot{y} + 2\,y = 0 \qquad (3.4.34)$$

Therefore it can be expected that every solution of (3.4.33) approaches the solution of (3.4.34), provided t_o is large enough. Thus:

$$y(t) \approx A\,e^{-(t-t_o)} + B\,e^{-2(t-t_o)} \qquad \text{for } t_o \text{ large.} \quad (3.4.35)$$

The question which will be dealt with here modifies (3.4.35) to the following problem:

> How large must t_o be chosen so that every solution of (3.4.33) satisfies the following requirement:
>
> $$|y(t)| \leq C\,e^{\delta(t-t_o)} \qquad (3.4.36)$$
>
> where the constant $\delta > -1$.

For this example, let δ be equal to -0.5 so that (3.4.36) can be specified as:

$$|y(t)| \leq C\, e^{-0.5(t-t_o)} \tag{3.4.37}$$

The solution of the last problem starts in the usual manner. Equation (3.4.33) is decomposed as follows:

$$\ddot{y} + 3\,\dot{y} + 2\,y = 6\,e^{-t}\,\dot{y} + 4\,e^{-t}\,y \tag{3.4.38}$$

The LHS of (3.4.38) has the following general solution:

$$y_{LHS} = y_o[2\,e^{-(t-t_o)} - e^{-2(t-t_o)}] + \dot{y}_o[e^{-(t-t_o)} - e^{-2(t-t_o)}] \tag{3.4.39}$$

Consequently:

$$g(t-u) = e^{-(t-u)} - e^{-2(t-u)}$$

so that (3.4.38) can be rewritten as:

$$y(t) = y_{LHS} + \int_{t_o}^{t} [6\,e^{-u}\,\dot{y}(u) + 4\,e^{-u}\,y(u)]\,g(t-u)\,du$$

or

$$y(t) = f(t) + \int_{t_o}^{t} k(t,u)\,y(u)\,du \tag{3.4.40}$$

with

$$\begin{aligned} f(t) &= e^{-(t-t_o)}\,[2\,y_o - 6\,e^{-t_o}\,y_o + \dot{y}_o] \\ &\quad + e^{-2(t-t_o)}\,[6\,e^{-t_o}\,y_o - y_o - \dot{y}_o] \\ &= f_o\,e^{-(t-t_o)} + \dot{f}_o\,e^{-2(t-t_o)} \end{aligned}$$

and

$$k(t,u) = e^{-(t-u)}\ 4\ e^{-u} + e^{-2(t-u)}\ 2\ e^{-u}$$

As in the case of a multiple root, it is convenient to define:

$$y(t)\ e^{(t-t_o)} = z(t) \qquad (3.4.41)$$

so that (3.4.40) becomes:

$$z(t) = f_o + \dot{f}_o\ e^{-(t-t_o)} + \int_{t_o}^{t} [4\ e^{-u} + 2\ e^{-u}\ e^{-(t-u)}] z(u)\ du \qquad (3.4.42)$$

Taking the modulus of (3.4.42)

$$|z(t)| \leq |f_o| + |\dot{f}_o|\ e^{-(t-t_o)} + \\ + \int_{t_o}^{t} [4\ e^{-(u-t_o)}\ e^{-t_o} + 2\ e^{-t_o}\ e^{-(u-t_o)} e^{-(t-u)}] |z(u)|\ du$$

and observing that:

$$e^{-(u-t_o)} \leq 1 \quad \text{for} \quad u \geq t_o$$

the following inequality is obtained:

$$|z(t)| \leq |f_o| + |\dot{f}_o|\ e^{-(t-t_o)} + \int_{t_o}^{t} [2\ m_o + m_o\ e^{-(t-u)}]\ |z(u)|\ du \qquad (3.4.43)$$

with

$$m_o = 2\, e^{-t_o}$$

Utilizing Theorem I gives:

$$|z(t)| \le Z(t)$$

where

$$Z(t) = |f_o| + |\dot{f}_o|\, e^{-(t-t_o)} + m_o \int_{t_o}^{t} [2 + e^{-(t-u)}] Z(u)\, du \tag{3.4.44}$$

The integral equation (3.4.44) can be solved by differentiating twice in order to get:

$$\ddot{Z} - (3\, m_o - 1)\, \dot{Z} - 2\, m_o\, Z = 0 \tag{3.4.45}$$

with:

$$Z(t_o) = |f_o| + |\dot{f}_o| \quad \text{and} \quad \dot{Z}(t_o) = 3\, m_o\, |f_o| + (3\, m_o - 1)\, |\dot{f}_o|$$

The characteristic equation of (3.4.45) has the following solutions:

$$\lambda_{1.2} = \frac{1}{2} \left(3\, m_o - 1 \pm \sqrt{9\, m_o^2 + 2\, m_o + 1}\right) \tag{3.4.46}$$

so that

$$Z(t) = C_1\, e^{\lambda_1 (t-t_o)} + C_2\, e^{\lambda_2 (t-t_o)}$$

where C_1 and C_2 can be determined from the initial conditions in (3.4.45). Recalling (3.4.41) and (3.4.44) gives:

$$|y(t)| \leq C_1 \, e^{(\lambda_1 - 1)(t - t_o)} + C_2 \, e^{(\lambda_2 - 1)(t - t_o)} \qquad (3.4.47)$$

If (3.4.47) is compared with (3.4.37) it follows:

$$(\lambda_1 - 1) \leq -0.5 \qquad \text{or} \qquad \lambda_1 \leq 0.5 \qquad (3.4.48)$$

Substituting (3.4.48) into (3.4.46) and observing (3.4.43) gives:

$$m_o = 2 \, e^{-t_o} \leq \frac{3}{14} \qquad \text{or} \qquad t_o \geq 2.24 \qquad (3.4.49)$$

Inequality (3.4.49) is a sufficient condition that (3.4.37) is satisfied.

CHAPTER IV

NONLINEAR SYSTEMS

4.1 Introduction

Various aspects of linear time-varying systems have been dealt with in Chapter II and Chapter III. The results will now be applied to the original nonlinear system (1.2.1). The objective of this chapter is to find a sector for $\phi(x,t)$ such that the solution of (1.2.1) is stable in the bounded input-bounded output sense.

In section 4.2 it will be shown that Theorem I can be applied to time-invariant nonlinear systems. A special example will indicate a class of systems, for which necessary and sufficient stability conditions might be obtained.

In section 4.3 the results from chapter III will be used to obtain sufficient stability conditions for a time-varying nonlinear system.

In section 4.4 two integral equations will be combined in order to generate successive approximations for the solution of a non-linear time-varying system. The proof of convergence will be sketched briefly.

4.2 Nonlinear Time-invariant System

Consider the system shown in Figure 4.A.

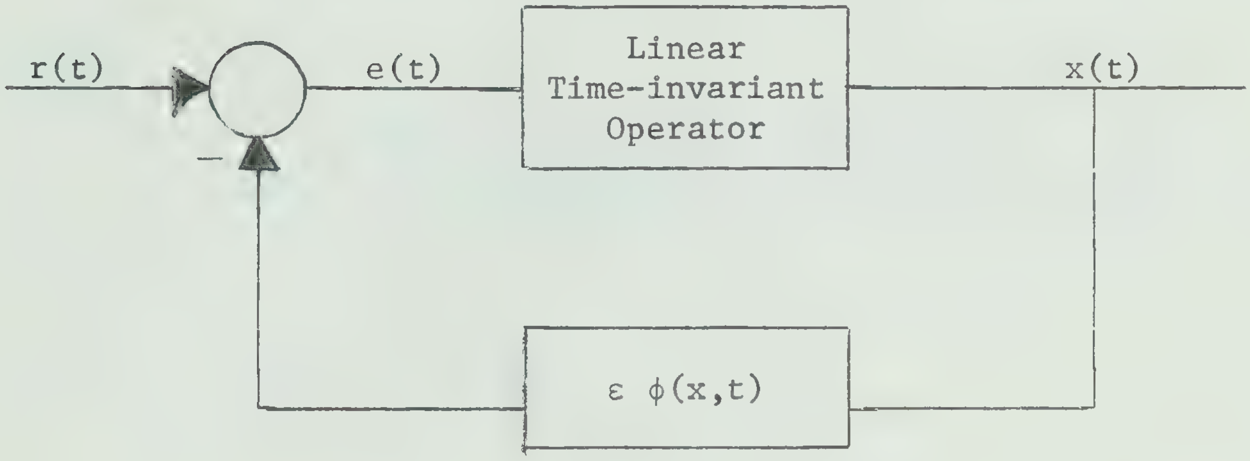


Figure 4.A

The time-invariant linear operator in the forward path is of the following form:

$$L(p)\ x(t) = L_r(p)\ e(t) \qquad (4.2.1)$$

where

$$L(p) = p^n + a_{n-1}p^{n-1} + \cdots + a_1 p + a_o$$

and

$$L_r(p) = m_r p^r + m_{r-1}p^{r-1} + \cdots + m_1 p + m_o$$

with

$$r < n \quad \text{and} \quad p^k = \frac{d^k}{dt^k}$$

As shown by Christensen and Trott [15], (4.2.1) is equivalent to the following integral equation:

$$x(t) = f(t) - \varepsilon\int_0^t h(t-\tau)\ \phi(x,\tau)\ d\tau \tag{4.2.2}$$

where

$$H(s) = \frac{m_r s^r + m_{r-1}s^{r-1} + \cdots + m_1 s + m_o}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_o} = \frac{N(s)}{D(s)} \tag{4.2.3}$$

$$h(t) = L^{-1}\{H(s)\}$$

$$e(t) = r(t) - \varepsilon\ \phi(x,t)$$

$$\begin{aligned}
D(s)\ L\{f(t)\} = \quad & x(0)\ [a_1 + a_2 s + \cdots + a_{n-1}s^{n-2} + s^{n-1}] \\
+ \quad & \dot{x}(0)\ [\ 0 + a_2 + \cdots + a_{n-1}s^{n-3} + s^{n-2}] \\
& \cdot \\
& \cdot \\
& \cdot \\
+ \quad & x^{(n-2)}(0)\ [\ 0 + 0 + \cdots + a_{n-1} + s\ ] \\
+ \quad & x^{(n-1)}(0)\ [\ 0 + 0 + \cdots + 0 + 1] \\
- \quad & e(0)\ [m_1 + m_2 s + \cdots + m_{r-1}s^{r-2} + m_r s^{r-1}] \\
- \quad & \dot{e}(0)\ [\ 0 + m_2 + \cdots + m_{r-1}s^{r-3} + m_r s^{r-2}] \\
& \cdot \\
& \cdot \\
& \cdot \\
- \quad & e^{(r-2)}(0)\ [\ 0 + 0 + \cdots + m_{r-1} + m_r s\ ] \\
- \quad & e^{(r-1)}(0)\ [\ 0 + 0 + \cdots + 0 + m_r\ ] \\
+ \quad & N(s)\ R(s)
\end{aligned}$$

Pole-shifting

Suppose (4.2.2) is rewritten in the following form:

$$x(t) = f_1(t) + \int_o^t h_1(t-\tau)\ [K\ x - \varepsilon\ \phi(x,\tau)]\ dt \qquad (4.2.5)$$

where K is a constant.

The unknown functions $f_1(t)$ and $h_1(t)$ can be determined in the following way:

Taking the Laplace transform of (4.2.2)

$$X(s) = F(s) - H(s)\ \varepsilon L\ \{\phi(x,t)\} \qquad (4.2.6)$$

and (4.2.5)

$$X(s) = F_1(s) + H_1(s)\ [K\ X(s) - \varepsilon\ L\{\phi(x,t)\}]$$

or

$$X(s) = \frac{F_1(s)}{1 - K\ H_1(s)} - \frac{H_1(s)}{1 - K\ H_1(s)}\ \varepsilon\ L\{\phi(x,t)\} \qquad (4.2.7)$$

and comparing (4.2.7) and (4.2.6) gives:

$$F(s) = \frac{F_1(s)}{1 - K\ H_1(s)} \qquad \text{and} \qquad H(s) = \frac{H_1(s)}{1 - K\ H_1(s)}$$

which implies:

$$F_1(s) = \frac{F(s)}{1 + K\ H(s)} \qquad \text{and} \qquad H_1(s) = \frac{H(s)}{1 + K\ H(s)}$$

Therefore $f_1(t)$ and $h_1(t)$ are given by:

$$f_1(t) = L^{-1}\{\frac{F(s)}{1 + K\,H(s)}\} \quad \text{and} \quad h_1(t) = L^{-1}\{\frac{H(s)}{1 + K\,H(s)}\} \qquad (4.2.8)$$

It is interesting to note that the poles of $H_1(s)$ can be investigated by the customary root locus method.

Assumption on the nonlinearity

For later purposes it has to be required that the nonlinearity in (4.2.2) satisfies the following condition:

$$\phi(0,t) = 0 \quad \text{for all} \quad t\,. \qquad (4.2.9)$$

After these preliminaries have been dealt with, the actual objective of this section can be approached now. To state it in general: A sector for $\phi(x,t)$ in (4.2.2) will be determined so that the solution of (4.2.2) is stable in the BIBO sense. The sector will be obtained in the following form:

$$K_1 \leq \frac{\phi(x,t)}{x} \leq K_2 \quad \text{with} \quad \begin{array}{l} K_1 = \text{const.} \\ K_2 = \text{const.} \end{array}$$

Outline of the method

In (4.2.5), $x(t)$ was defined by the following integral equation:

$$x(t) = f_1(t) + \int_0^t h_1(t-\tau)\,[K\,x - \varepsilon\,\phi(x,\tau)]\,d\tau\,.$$

Therefore $|x(t)|$ satisfies the following integral inequality:

$$|x(t)| \leq |f_1(t)| + \int_o^t |h_1(t-\tau)|\,|x(\tau)|\,\left|K - \varepsilon \frac{\phi(x,\tau)}{x(\tau)}\right| d\tau \tag{4.2.10}$$

Letting

$$\left|K - \varepsilon \frac{\phi(x,\tau)}{x(t)}\right| \leq m_o \tag{4.2.11}$$

and applying Theorem I to (4.2.10) gives:

$$|x(t)| \leq X(t)$$

where

$$X(t) = F_1(t) + \int_o^t H_1(t-\tau)\, m_o\, X(\tau)\, d\tau$$

with

$$F_1(t) \geq |f_1(t)| \qquad \text{and} \qquad H_1(t-\tau) \geq |h_1(t-\tau)|$$

Similarly as in Chapter III it will be found that m_o has to be smaller than some positive constant δ in order that $X(t)$ and consequently $|x(t)|$ is bounded.

Thus

$$m_o \leq \delta$$

or because of (4.2.11)

$$\left|K - \varepsilon \frac{\phi(x,t)}{x(t)}\right| \leq \delta$$

which implies:

$$K - \delta \leq \varepsilon \frac{\phi(x,t)}{x(t)} \leq K + \delta \qquad (4.2.12)$$

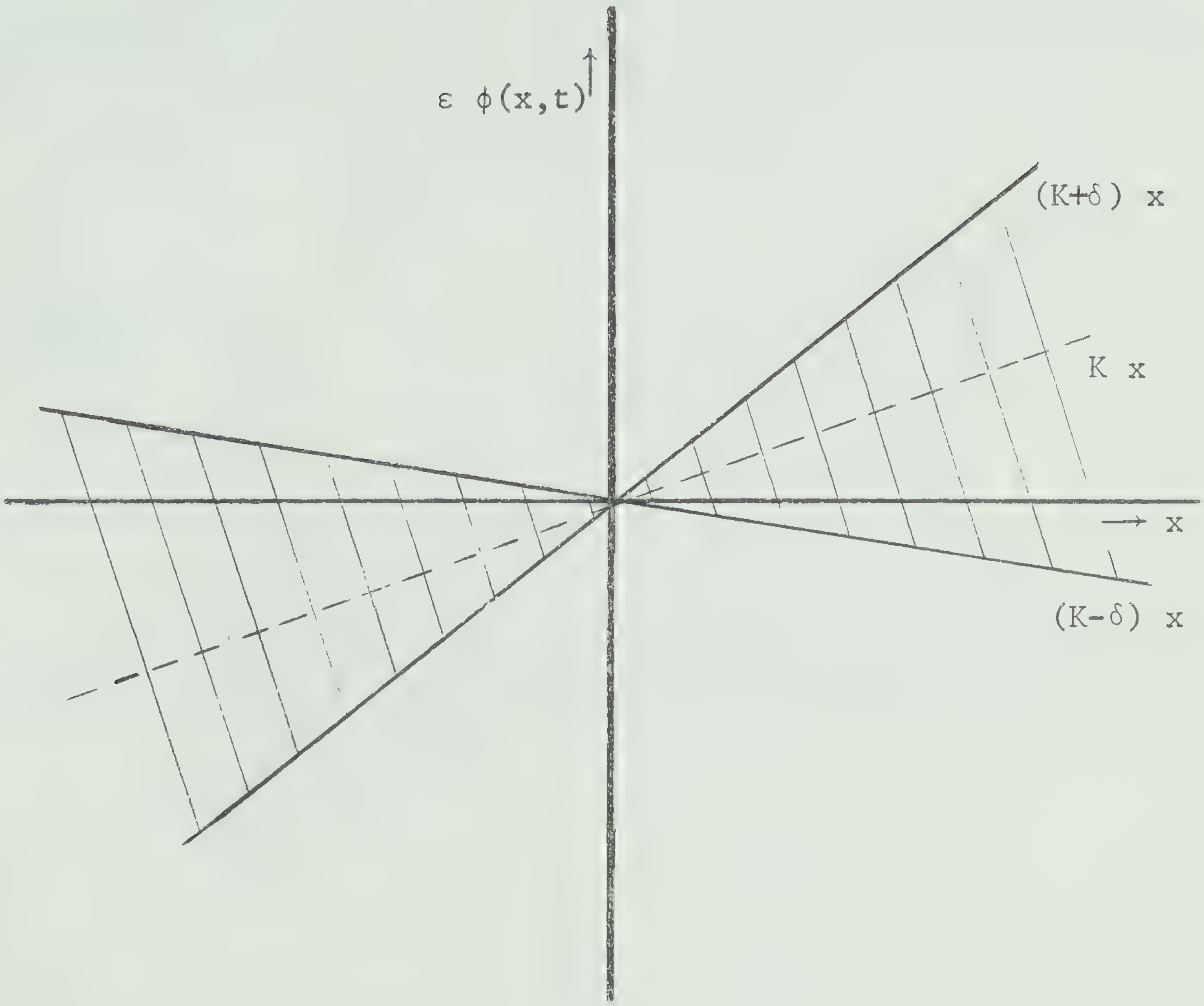


Figure 4.B

Inequality (4.2.12) is illustrated in Figure 4.B. The shaded area depicts an admissible region for $\varepsilon\,\phi(x,t)$ which ensures that the solution of (4.2.2) is bounded by at least a constant.

In general, δ is a function of K. Therefore the admissible region for $\varepsilon\,\phi(x,t)$ can be represented as a function of K, as shown in Figure 4.C, for example.

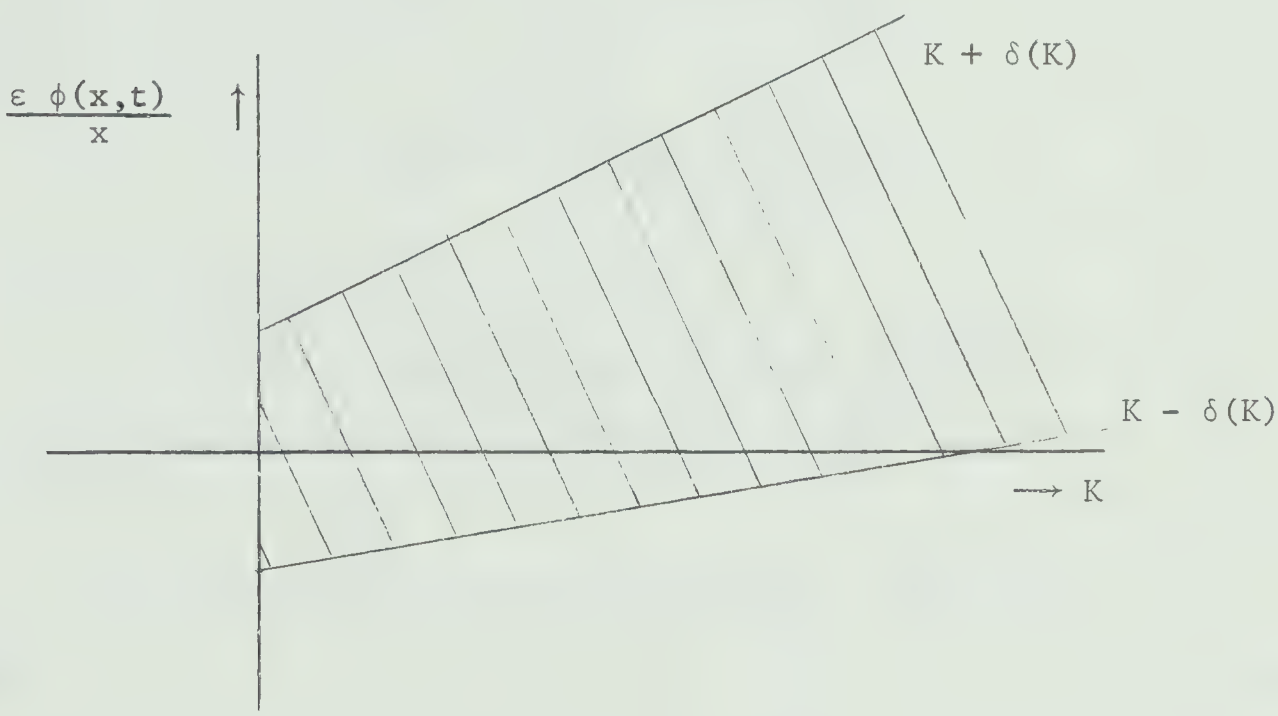


Figure 4.C

Example

Suppose the following system is given:

$$\ddot{x} + 2\,\dot{x} + x + \varepsilon\,\phi(x,t) = 0 \qquad \begin{matrix} x(0) = x_o \\ \dot{x}(0) = \dot{x}_o \end{matrix} \tag{4.2.13}$$

Substituting the appropriate values into the following equations gives for

(4.2.3):
$$H(s) = \frac{1}{s^2 + 2\,s + 1}$$

(4.2.4):
$$[s^2 + 2\,s + 1]\,F(s) = x_o(s + 2) + \dot{x}_o$$

(4.2.8):
$$F_1(s) = \frac{x_o(s + 2) + \dot{x}_o}{s^2 + 2\,s + 1 + K}$$

$$H_1(s) = \frac{1}{s^2 + 2\,s + 1 + K}$$

In order to have a simple notation, let $K = k^2$. Then the corresponding functions in the time domain become:

$$f_1(t) = \frac{1}{k}\,e^{-t}[x_o(\sin kt + k\cos kt) + \dot{x}_o \sin kt]$$

and (4.2.14)

$$h_1(t) = \frac{1}{k}\,e^{-t} \sin kt\ .$$

Utilizing (4.2.14), equation (4.2.5) can be written as:

$$x(t) = f_1(t) + \int_o^t \frac{1}{k}\,e^{-(t-\tau)} \sin k(t-\tau)\ [k^2x - \varepsilon\ \phi(x,\tau)]\ d\tau \tag{4.2.15}$$

Defining

$$z(t) = x(t)\ e^t \tag{4.2.16}$$

simplifies (4.2.15) to

$$z(t) = f_1(t)\, e^t + \int_o^t \frac{1}{k} \sin k(t-\tau)\; z(\tau)\; [k^2 - \varepsilon \frac{\phi(x,\tau)}{x}]\, d\tau \qquad (4.2.17)$$

Taking the modulus of (4.2.17) gives the following inequality:

$$|z(t)| \leq |f_1\, e^t| + \int_o^t \frac{1}{k}\, |\sin k(t-\tau)|\; |z(\tau)|\; |k^2 - \varepsilon \frac{\phi(x,\tau)}{x}|\, d\tau \qquad (4.2.18)$$

From (4.2.14) it follows:

$$|f_1(t)\, e^t| \leq \frac{1}{k}\, [|x_o|\, (1 + k) + |\dot{x}_o|] \leq M_o$$

so that (4.2.18) can be rewritten as:

$$|z(t)| \leq M_o + \int_o^t \frac{1}{k} |z(\tau)|\, m_o\, d\tau \qquad (4.2.19)$$

with

$$|k^2 - \varepsilon \frac{\phi(x,t)}{x}| \leq m_o \qquad (4.2.20)$$

Applying Theorem I to (4.2.19) results in:

$$|z(t)| \leq Z(t) \qquad (4.2.21)$$

where

$$Z(t) = M_o + \int_o^t \frac{1}{k}\, Z(\tau)\, m_o\, d\tau$$

or

$$Z(t) = M_o\, e^{(m_o/k)\, t}$$

which, observing (4.2.21) and (4.2.16), implies for $x(t)$:

$$|x(t)| \leq M_o \, e^{\left[\frac{m_o}{k} - 1\right] t} \tag{4.2.22}$$

In order that $|x(t)|$ is bounded by at least a constant, it is required that:

$$\frac{m_o}{k} \leq 1$$

or because of (4.2.20)

$$\left|k^2 - \varepsilon \, \frac{\phi(x,t)}{x}\right| \leq k$$

which is equivalent to:

$$k^2 - k \leq \varepsilon \, \frac{\phi(x,t)}{x} \leq k^2 + k \tag{4.2.23}$$

In Fig. 4.D, inequality (4.2.23) is plotted with respect to k. For $k = 1$, for example, the admissible sector for the nonlinearity is shown in Fig. 4.E.

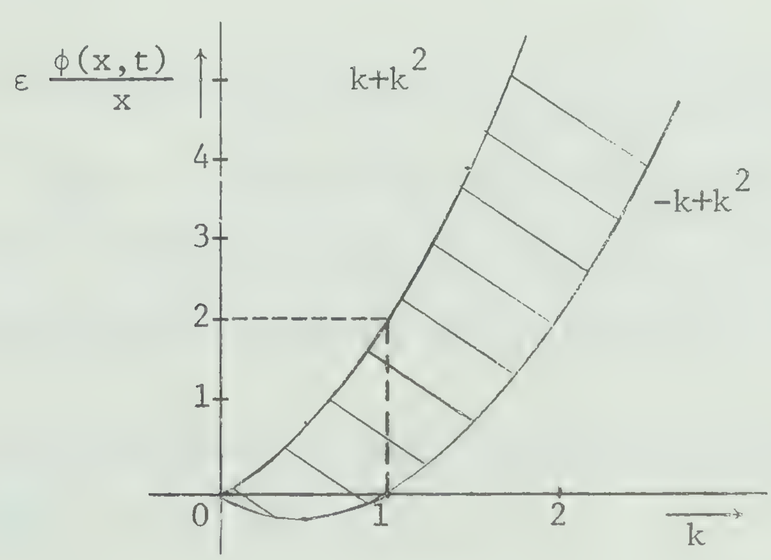


Figure 4.D

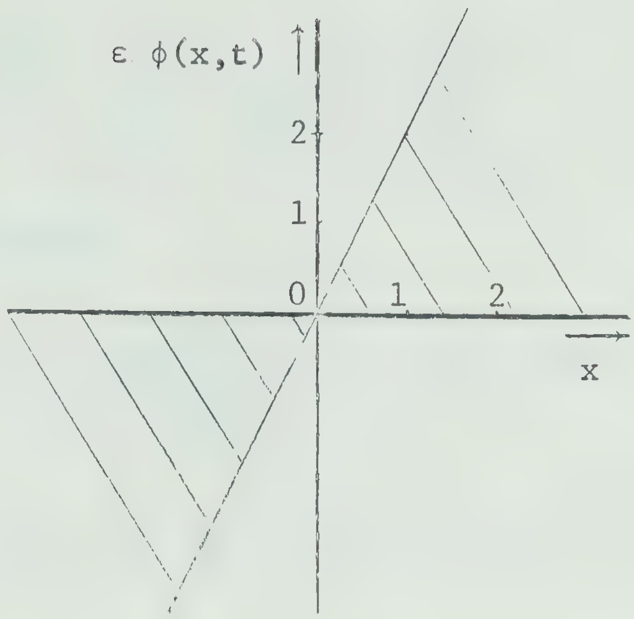


Figure 4.E

In (4.2.19), the modulus of the kernel was approximated in the following way:

$$|\sin k(t-\tau)| \leq 1$$

Therefore (4.2.23) is only a sufficient stability condition for system (4.2.13).

It would seem logical that for positive kernels, that is $|h_1(t-\tau)| \equiv h_1(t-\tau)$, the method under consideration could give necessary and sufficient stability conditions. The necessity is ensured when (4.2.12) coincides with the Hurwitz sector.

This idea will be shown to be true for the following example. Suppose, $H(s)$ in (4.2.3) is given by:

$$H(s) = \frac{s + 2}{s^2 + 4\,s + 3}$$

so that (4.2.4) becomes:

$$F(s) = \frac{x_o(4 + s) + \dot{x}_o - e_o}{s^2 + 4\,s + 3} + H(s)\ R(s)$$

The functions $H_1(s)$ and $F_1(s)$ are determined from (4.2.8) as:

$$H_1(s) = \frac{s + 2}{s^2 + (4 + K)\ s + 3 + 2\ K}$$

and

$$F_1(s) = \frac{x_o(4 + s) + \dot{x}_o - e_o + (s + 2)\ R(s)}{s^2 + (4 + k)\ s + 3 + 2\ K}$$

Expanding $H_1(s)$ and $F_1(s)$ into partial fractions gives:

$$H_1(s) = [\frac{s_1 + 2}{s_1 - s_2}][\frac{1}{s - s_1}] - [\frac{s_2 + 2}{s_1 - s_2}]\ [\frac{1}{s - s_2}]$$

and

$$F_1(s) = [\frac{x_o\ (4 + s_1) + \dot{x}_o - e_o}{s_1 - s_2}][\frac{1}{s - s_1}]$$

$$-[\frac{x_o\ (s_2 + 4) + \dot{x}_o - e_o}{s_1 - s_2}][\frac{1}{s - s_2}] + H_1(s)\ R(s)$$

with

$$s_1 = -\frac{1}{2}\ [(4 + K) - \sqrt{4 + K^2}\ ]$$

and (4.2.24)

$$s_2 = -\frac{1}{2}\ [(4 + K) + \sqrt{4 + K^2}\ ]$$

The corresponding functions in the time domain are:

$$h_1(t) = S_1 e^{s_1 t} + S_2 e^{s_2 t} \tag{4.2.25}$$

where

$$S_1 = \frac{1}{2}\left[1 - \frac{K}{\sqrt{4+K^2}}\right] , \quad S_2 = \frac{1}{2}\left[1 + \frac{K}{\sqrt{4+K^2}}\right] \tag{4.2.26}$$

and

$$f_1(t) = \frac{x_o[2 - \frac{1}{2}K + \frac{1}{2}\sqrt{4+K^2}\,] + \dot{x}_o - e_o}{\sqrt{4+K^2}}\, e^{s_1 t}$$

$$- \frac{x_o[2 - \frac{1}{2}K - \frac{1}{2}\sqrt{4+K^2}\,] + \dot{x}_o - e_o}{\sqrt{4+K^2}}\, e^{s_2 t}$$

$$+ \int_o^t h_1(t-\tau)\, r(\tau)\, d\tau$$

Since $s_2 \leq s_1 \leq -1$ and assuming $|r(t)| \leq R_o$, it can be verified that:

$$|f_1(t)| \leq \frac{2\,R_o}{s_1\,s_2} + e^{s_1 t}\,[6|x_o| + 2|\dot{x}_o| + 2|e_o|]$$

$$\leq \frac{2\,R_o}{s_1\,s_2} + e^{s_1 t}\,M_o \tag{4.2.27}$$

Now, every term in (4.2.5) has been estimated. From (4.2.25) it is evident that $h_1(t-\tau) \geq 0$ for all $K \geq 0$. Therefore, (4.2.10) becomes for this example:

$$|x(t)| \leq \frac{2R_o}{s_1 s_2} + e^{s_1 t} M_o + \int_o^t h_1(t-\tau) \; |x(\tau)| \; m_o \; d\tau \qquad (4.2.28)$$

where m_o is defined as in (4.2.11).

Applying Theorem I to (4.2.28) yields:

$$|x(t)| \leq X(t)$$

for

$$X(t) = \frac{2 \; R_o}{s_1 \; s_2} + e^{s_1 t} M_o + \int_o^t h_1(t-\tau) \; X(\tau) \; m_o \; d\tau \qquad (4.2.29)$$

As usual it is convenient to define

$$X(t) \; e^{-s_1 t} = Z(t) \qquad (4.2.30)$$

so that (4.2.29) becomes

$$Z(t) = \frac{2 \; R_o}{s_1 \; s_2} e^{-s_1 t} + M_o + \int_o^t [S_1 + S_2 \; e^{-(s_1-s_2)(t-\tau)}] m_o \; Z(\tau) \; d\tau \qquad (4.2.31)$$

Differentiating (4.2.31) twice and eliminating the integral term gives:

$$\ddot{Z} + \dot{Z}[-(s_2-s_1) - (S_1+S_2) \; m_o] + Z[S_1 \; m_o \; (s_2-s_1)] = 2 \; R_o \; e^{-s_1 t} \qquad (4.2.32)$$

Recalling (4.2.24) and (4.2.26), the coefficients of the LHS of (4.2.32) are evaluated as:

$$-(s_2 - s_1) - (S_1 + S_2)\, m_o = \sqrt{4 + K^2} - m_o$$

$$S_1\ (s_2 - s_1)\, m_o = \frac{1}{2} m_o\, [K - \sqrt{4 + K^2}\,]$$

Equation (4.2.32) has the following characteristic roots:

$$\lambda_{1,2} = \frac{1}{2}\, [m_o - \sqrt{4 + K^2} \pm \sqrt{4 + (K - m_o)^2}\,] \qquad (4.2.33)$$

Consequently, the general solution of (4.2.32) is:

$$Z = A\, e^{\lambda_1 t} + B\, e^{\lambda_2 t} + \int_o^t [C_1\, e^{\lambda_1 (t-\tau)} + C_2\, e^{\lambda_2 (t-\tau)}]\, 2\, R_o\, e^{-s_1 \tau}\, d\tau$$

which, because of (4.2.30), means in terms of $x(t)$:

$$|x(t)| \leq A\, e^{(\lambda_1 + s_1) t} + B\, e^{(\lambda_2 + s_1) t} +$$

$$+ \int_o^t [C_1\, e^{(\lambda_1 + s_1)(t-\tau)} + C_2\, e^{(\lambda_2 + s_1)(t-\tau)}]\, 2 R_o\, d\tau$$

In order that $x(t)$ is bounded by a decreasing exponential, it is required that:

$$\lambda_1 + s_1 < 0$$

or, substituting (4.2.24) and (4.2.33) for s_1 and λ_1 respectively:

$$\frac{1}{2}\, [m_o - \sqrt{4 + K^2} + \sqrt{4 + (K - m_o)^2}\,] < \frac{1}{2}\, [4 + K - \sqrt{4 + K^2}\,]$$

or

$$4 + (K - m_o)^2 < (4 + K - m_o)^2$$

or

$$- 12 < 8 \ (K - m_o)$$

or

$$m_o < 1.5 + K \qquad (4.2.34)$$

Combining (4.2.11) and (4.2.34) gives:

$$- 1.5 < \varepsilon \frac{\phi(x,t)}{x} < 1.5 + 2 K \qquad (4.2.35)$$

Inequality (4.2.35) is the desired result. Since $K > 0$ is arbitrary, (4.2.35) can be extended to the Hurwitz sector $(-1.5,\infty)$.

4.3 Nonlinear Time-Varying System

Suppose the time-invariant linear operator in Fig. 4.A is replaced by the following time-varying linear operator:

$$L(p,t) \ x(t) = L_r(p,t) \ e(t) \qquad (4.3.1)$$

where

$$L(p,t) = p^n + a_{n-1}(t) \ p^{n-1} + \cdots + a_1(t) \ p + a_o(t)$$

$$(4.3.2)$$

$$L_r(p,t) = m_r(t) \ p^r + m_{r-1}(t) \ p^{r-1} + \cdots + m_1(t) \ p + m_o(t)$$

with

$$e(t) = r(t) - \varepsilon \ \phi(x,t) \ , \ p^k = \frac{d^k}{dt^k} \ , \quad \text{and} \quad r < n$$

First, assume that the LHS of (4.3.1) has a closed form solution. Then, by means of convolution, (4.3.1) can be written as:

$$x(t) = y(t) + \int_0^t h(t,\tau)\{[L_r(p,t)\ e(t)\Big|_{t=\tau}\}\ d\tau \tag{4.3.3}$$

where $y(t)$ is the solution of:

$$L(p,t)\ y(t) = 0 \tag{4.3.4}$$

subject to the given initial conditions and $h(t,\tau)$ is the appropriate impulse response.

If (4.3.3) is integrated by parts, one gets an equation of the following form:

$$x(t) = f(t) + \int_0^t h(t,\tau)\ r(\tau)\ d\tau - \varepsilon\int_0^t h(t,\tau)\ \phi(x,\tau)\ d\tau \tag{4.3.5}$$

From (4.3.5) onward, the problem is dealt with in a way similar to that used for the time-invariant case.

Taking the modulus of (4.3.5)

$$|x(t)| \leq |f(t)| + \int_0^t |h(t,\tau)|\ |r(\tau)|\ d\tau + \varepsilon\int_0^t |h(t,\tau)|\ |\phi(x,\tau)|\ d\tau \tag{4.3.6}$$

and assuming:

$$|f(t)| + \int_0^t |h(t,\tau)|\ |r(\tau)|\ d\tau \leq M_o$$

inequality (4.3.6) becomes:

$$|x(t)| \leq M_o + \int_o^t |h(t,\tau)|\ |x(\tau)|\ |\varepsilon \frac{\phi(x,\tau)}{x}|\ d\tau \qquad (4.3.7)$$

Theorem I will be applied to (4.3.7) in order to get an admissible region for $\phi(x,t)$.

Example

$$(t+1)^2\ \ddot{x} + 4\ (t+1)\ \dot{x} + 2\ x + \varepsilon\ \phi(x,t) = 0$$

or

$$\ddot{x} + \frac{4}{t+1}\ \dot{x} + \frac{2}{(t+1)^2}\ x = -\ \varepsilon\ \frac{\phi(x,t)}{(t+1)^2} \qquad (4.3.8)$$

The LHS of (4.3.8) is identical to (2.3.7a) so that (2.3.7) and (2.3.8) can be used to obtain:

$$x(t) = x(0)\ [1 + \frac{t}{t+1}]\ \frac{1}{t+1} + \dot{x}(0)\ \frac{t}{(t+1)^2}$$

$$- \varepsilon \int_o^t \frac{(\tau+1)^2}{(t+1)^2}\ (t-\tau)\ \frac{\phi(x,\tau)}{(\tau+1)^2}\ d\tau \qquad (4.3.9)$$

Taking the modulus of (4.3.9) gives:

$$|x(t)| \leq |x(0)|\ \frac{2}{t+1} + |\dot{x}(0)|\ \frac{t}{(t+1)^2} \qquad (4.3.10)$$

$$+ \int_o^t \frac{t-\tau}{(t+1)^2}\ \varepsilon |\phi(x,\tau)|\ d\tau$$

From Theorem I it follows:

$$|x(t)| \leq X(t) \tag{4.3.11}$$

where

$$X(t)(t+1)^2 = |x(0)|\, 2\,(t+1) + |\dot{x}(0)|\,t + \int_0^t (t-\tau)\, X(\tau)\, m_o\, d\tau \tag{4.3.12}$$

with

$$m_o \geq \varepsilon \left|\frac{\phi(x,t)}{x}\right| \tag{4.3.13}$$

If (4.3.12) is differentiated twice, one gets:

$$\ddot{X}\,(t+1)^2 + 4\,\dot{X}\,(t+1) + 2\,X = X\,m_o$$

or

$$\ddot{X} + \frac{4}{t+1}\,\dot{X} + \frac{2-m_o}{(t+1)^2}\,X = 0 \tag{4.3.14}$$

Equation (4.3.14) has the following solution:

$$X(t) = C_1\,(t+1)^{r_1} + C_2\,(t+1)^{r_2}$$

with

$$r_{1,2} = -\frac{3}{2} \pm \sqrt{m_o + \frac{1}{4}}$$

Consequently, the solution of (4.3.8) is stable if:

$$m_o + \frac{1}{4} \leq \frac{9}{4}$$

or because of (4.3.13):

$$-2 \leq \varepsilon\,\frac{\phi(x,t)}{x} \leq 2 \tag{4.3.15}$$

Inspection of (4.3.8) shows that the last result (4.3.15) can be anticipated. If an attempt is made to enlarge the admissible region for the nonlinearity by means of pole-shifting, it is found that this can be done successfully for the last example.

In general, however, the following difficulty will be encountered if the "pole-shifting" technique is to be carried over to a time-varying linear system.

Consider:

$$\ddot{x} + a_1(t)\ \dot{x} + a_o(t)\ x = -\ \varepsilon\ \phi(x,t) \tag{4.3.16}$$

and assume that the LHS of (4.3.16) has a closed form solution. After adding $K\ x(t)$ on both sides of (4.3.16), that is:

$$\ddot{x} + a_1(t)\ \dot{x} + a_o(t)\ x + K\ x = K\ x - \varepsilon\ \phi(x,t) \tag{4.3.17}$$

the LHS of (4.3.17) usually cannot be solved in closed form anymore. In this case, approximations have to be used which will be discussed in the remainder of this section.

Assume, the following system is given:

$$L(p,t)\ x(t) = r(t) - \varepsilon\ \phi(x,t) \tag{4.3.18}$$

where $L(p,t)$ is defined in (4.3.2). In general, no closed form solution is known for

$$L(p,t)\ y(t) = 0\ . \tag{4.3.19}$$

However, if (4.3.19) satisfies the Cauchy-Lipschitz condition as discussed in Chapter II, then the existence and uniqueness of $y(t)$ and consequently $h(t,\tau)$ are guaranteed. Consequently (4.3.18) can be written as:

$$x(t) = y(t) + \int_0^t h(t,\tau)\ r(\tau)\ d\tau - \varepsilon\int_0^t h(t,\tau)\ \phi(x,\tau)\ d\tau \tag{4.3.20}$$

Taking the modulus of (4.3.20) gives:

$$|x(t)| \leq |y(t)| + \int_0^t |h(t,\tau)|\ |r(\tau)|\ d\tau + \varepsilon\int_0^t |h(t,\tau)||\phi(x,\tau)|\, d\tau \tag{4.3.21}$$

Now, the approximations for $|y(t)|$ and $|h(t,\tau)|$, obtained in Chapter III, can be substituted into (4.3.21). The remaining part of the problem can be treated in the customary manner.

<u>Example</u>

Consider:

$$\ddot{x} + [2 - \tfrac{1}{3}\cos t]\ \dot{x} + [1 - \tfrac{2}{3}\cos t]\ x = -\ \varepsilon\ \phi(x,t) \tag{4.3.22}$$

The linear part of (4.3.22) was dealt with in example (3.4.1), so that the appropriate results can be employed here.

Recalling (3.4.10):

$$|h(t,\tau)| \leq \frac{6}{8.5}\ [e^{-0.125(t-\tau)} - e^{-1.54(t-\tau)}]$$

and (3.4.11)

$$|y(t)| \leq F_o \, e^{-0.125\, t}$$

inequality (4.3.21) can be written as:

$$|x(t)| \leq F_o \, e^{-0.125t} + \varepsilon \int_o^t \frac{6}{8.5} \, [e^{-0.125(t-\tau)} - e^{-1.54(t-\tau)}] \, |\phi(x,t)| \, d\tau \tag{4.3.23}$$

Applying Theorem I to (4.3.23) gives the following sufficient condition for stability:

$$\left|\varepsilon \, \frac{\phi(x,t)}{x}\right| \leq 0.19$$

4.4 Approximations

Suppose the following time-invariant nonlinear system is to be approximated on some interval $[0,T]$:

$$L(p) \, x(t) + \varepsilon \, \phi(x) = r(t) \tag{4.4.1}$$

A customary way to do this is to rewrite (4.4.1) as:

$$x(t) = y(t) + \int_o^t h(t-\tau) \, r(\tau) \, d\tau - \varepsilon \int_o^t h(t-\tau) \, \phi(x) \, d\tau \tag{4.4.2}$$

and to generate the following sequence from (4.4.2):

$$x_o(t) = y(t) + \int_o^t h(t-\tau) \, r(\tau) \, d\tau$$

$$x_1(t) = x_o(t) - \varepsilon \int_o^t h(t-\tau) \, \phi(x_o) \, d\tau$$

or in general

$$x_{n+1}(t) = x_o(t) - \varepsilon\int_o^t h(t-\tau)\ \phi(x_n)\ d\tau \qquad (4.4.3)$$

Trott and Christensen [3] show that (4.4.3) is a convergent sequence on $[0,T]$ if $\phi(x)$ satisfies a Lipschitz condition.

A similar approach can be used for the following time-varying nonlinear system:

$$L(p,t)\ x(t) + \varepsilon\ \phi(x,t) = r(t) \qquad (4.4.4)$$

where $L(p,t)$ is defined in (4.3.2).

The equivalent integral equation for (4.4.4) is given in (4.3.5). Thus:

$$x(t) = y(t) + \int_o^t h(t,\tau)\ r(\tau)\ d\tau - \varepsilon\int_o^t h(t,\tau)\ \phi(x,\tau) d\tau \quad (4.4.5)$$

In general, $y(t)$ and $h(t,\tau)$ cannot be obtained in closed form but are given by integral equations. This was shown in Chapter III. Substituting (3.2.11)

$$y(t) = f(t) + \int_o^t k(t,u)\ y(u)\ du \qquad (4.4.6)$$

and (3.2.15)

$$h(t,\tau) = g(t,\tau) + \int_\tau^t k(t,u)\ h(u,\tau)\ du \qquad (4.4.7)$$

into (4.4.5) gives:

$$x(t) = f(t) + \int_0^t k(t,u)\ y(u)\ du$$
$$+ \int_0^t [g(t,\tau) + \int_\tau^t k(t,u)\ h(u,\tau)\ du][r(\tau) - \varepsilon\ \phi(x,\tau)]\ d\tau$$

or

$$x(t) = f(t) + \int_0^t g(t,\tau)\ [r(\tau) - \varepsilon\ \phi(x,\tau)]\ d\tau + \int_0^t k(t,u)\ y(u)\ du$$
$$+ \int_0^t \int_\tau^t k(t,u)\ h(u,\tau)\ du\ [r(\tau) - \varepsilon\ \phi(x,\tau)]\ d\tau \qquad (4.4.8)$$

If the terms under the double integral are bounded on $[0,T]$, which is true for most physical systems, then, by Fubini's Theorem [16], the order of integration can be interchanged. Therefore the double integral becomes:

$$\int_0^t k(t,u) \int_{\tau=0}^u h(u,\tau)\ [r(\tau) - \varepsilon\ \phi(x,\tau)]\ d\tau\ du$$

or because of (4.4.5)

$$\int_0^t k(t,u)\ [x(u) - y(u)]\ du \qquad (4.4.9)$$

Substituting (4.4.9) into (4.4.8) gives:

$$x(t) = f(t) + \int_0^t g(t,\tau)\ r(\tau)\ d\tau$$
$$+ \int_0^t k(t,\tau)\ x(\tau)\ d\tau - \varepsilon \int_0^t g(t,\tau)\ \phi(x,\tau)\ d\tau \qquad (4.4.10)$$

Equation (4.4.10) includes several special cases.

(a) Linear time-invariant system $[\phi(x,t) \equiv 0]$

For this case, (4.4.6) and (4.4.7) can be simplified to:

$$y(t) \equiv f(t) \quad , \quad k(t,u) \equiv 0$$

$$h(t,\tau) \doteq g(t-\tau)$$

Therefore, (4.4.10) becomes:

$$x(t) = f(t) + \int_0^t g(t-\tau)\, r(\tau)\, d\tau$$

(b) Nonlinear time-invariant system

The functions $y(t)$ and $h(t,\tau)$ are the same as under (a). Consequently:

$$x(t) = f(t) + \int_0^t g(t-\tau)\, r(\tau)\, d\tau - \varepsilon \int_0^t g(t-\tau)\, \phi(x,\tau)\, d\tau$$

(c) Homogeneous linear system $[r(t) \equiv 0 \ , \ \phi(x,t) \equiv 0]$

For these assumptions, (4.4.10) is identical to (4.4.6)

From (4.4.10) the following sequence of successive approximations can be generated:

$$x_o(t) = f(t) + \int_0^t g(t,\tau)\, r(\tau)\, d\tau \quad ,$$

$$x_1(t) = x_o(t) + \int_0^t k(t,\tau)\, x_o(\tau)\, d\tau - \varepsilon \int_0^t g(t,\tau)\, \phi(x_o,\tau)\, d\tau \quad ,$$

or in general

$$x_{n+1}(t) = x_o(t) + \int_o^t k(t,\tau)\; x_n(\tau)\; d\tau - \varepsilon\int_o^t g(t,\tau)\; \phi(x_n,\tau)\; d\tau \tag{4.4.11}$$

A sufficient condition, that $\{x_n\}_{n=0}^{\infty}$ is a convergent sequence, is:

$$\left|\frac{\phi(x_1,t) - \phi(x_2,t)}{x_1 - x_2}\right| \leq M = \text{const} \quad \text{for all} \quad t \in [0,T]. \tag{4.4.12}$$

The proof is similar to the one given by Trott and Christensen [3] for time-invariant systems. To sketch it briefly:

Define

$$\begin{aligned} x(t)\; e^{-\lambda t} &= z(t) \qquad \text{for} \quad \lambda = \text{positive integer} \\ x_n(t)\; e^{-\lambda t} &= z_n(t) \end{aligned} \tag{4.4.13}$$

so that (4.4.11) becomes:

$$\begin{aligned} z_{n+1}(t) = x_o(t)\; e^{-\lambda t} &+ \int_o^t k(t,\tau)\; e^{-\lambda t}\; e^{\lambda\tau}\; z_n(\tau)\; d\tau \\ &-\varepsilon\int_o^t g(t,\tau)\; e^{-\lambda t}\; \phi(x_n,\tau)\; d\tau \end{aligned}$$

For the difference of two successive terms one gets:

$$\begin{aligned} |z_{n+1} - z_n| \leq \int_o^t |k(t,\tau)|\; e^{-\lambda(t-\tau)}\; |z_n(\tau) - z_{n-1}(\tau)|\; d\tau \; + \\ + \varepsilon\int_o^t |g(t,\tau)|\; e^{-\lambda(t-\tau)}\; |z_n(\tau) - z_{n-1}(\tau)|\; \left|\frac{\phi(x_n,\tau) - \phi(x_{n-1},\tau)}{x_n - x_{n-1}}\right|\; d\tau \end{aligned}$$

Utilizing assumption (4.4.12) gives:

$$\sup_{[0,T]} \{|z_{n+1}(t) - z_n(t)|\} \leq \sup_{[0,T]} \{|z_n(t) - z_{n-1}(t)|\} \frac{H\ (1 + \varepsilon M)}{\lambda} \tag{4.4.14}$$

with

$$H = \sup_{[0,T]} \{g(t,\tau)\ ,\ k(t,\tau)\}\quad ,\quad \text{for}\quad 0 \leq \tau \leq t \leq T$$

If λ is chosen sufficiently large, (4.4.14) satisfies the contraction condition, which implies that the following series is convergent.

$$z(t) = z_o(t) + z_1(t) - z_o(t) + z_2(t) - z_1(t) + \cdots$$

From (4.4.13) it follows that $\{x_n(t)\}_{n=1}^{\infty}$ is convergent too.

Example

Suppose, the first two terms of (4.4.11) are to be evaluated for the following system:

$$\ddot{x} + 2\ (1 - e^{-t})\ \dot{x} + x + |x| = e^{-t} \qquad \begin{array}{l} x(0) = 0 \\ \dot{x}(0) = 1 \end{array} \tag{4.4.15}$$

First, the appropriate functions $y(t)$ and $h(t,\tau)$ will be determined from (4.4.6) and (4.4.7) respectively. The linear homogeneous part of (4.4.15) is:

$$\ddot{y} + 2\ (1 - e^{-t})\ \dot{y} + y = 0 \qquad \begin{array}{l} y(0) = 0 \\ \dot{y}(0) = 1 \end{array}$$

or

$$\ddot{y} + 2\,\dot{y} + y = 2\,e^{-t}\,\dot{y} \tag{4.4.16}$$

Therefore

$$y(t) = t\,e^{-t} + \int_0^t 2\,(t-u)\,e^{-(t-u)}\,e^{-u}\,\dot{y}(u)\,du$$

or

$$y(t) = t\,e^{-t} + \int_0^t 2\,e^{-t}\,y(u)\,du$$

Similarly, one gets for (4.4.7)

$$h(t,\tau) = (t-\tau)\,e^{-(t-\tau)} + \int_\tau^t 2\,e^{-t}\,h(u,\tau)\,du$$

A comparison with (4.4.6) and (4.4.7) shows that:

$$\begin{aligned} f(t) &= t\,e^{-t} \\ k(t,u) &= 2\,e^{-t} \\ g(t,\tau) &= (t-\tau)\,e^{-(t-\tau)} \end{aligned} \tag{4.4.17}$$

Substituting (4.4.17) into (4.4.10) gives:

$$\begin{aligned} x(t) = t\,e^{-t} &+ \int_0^t (t-\tau)\,e^{-(t-\tau)}\,e^{-\tau}\,d\tau \\ &+ \int_0^t 2\,e^{-t}\,x(\tau)\,d\tau - \int_0^t (t-\tau)\,e^{-(t-\tau)}\,|x(\tau)|\,d\tau \end{aligned}$$

According to (4.4.11), $x_o(t)$ becomes:

$$x_o(t) = t\, e^{-t} + \int_o^t (t-\tau)\, e^{-t}\, d\tau = e^{-t}\, [t + \frac{t^2}{2}]$$

Iterating once generates $x_1(t)$ as:

$$x_1(t) = x_o(t) + \int_o^t 2\, e^{-t}\, e^{-\tau}\, [\tau + \frac{\tau^2}{2}]\, d\tau$$

$$- \int_o^t (t-\tau)\, e^{-(t-\tau)}\, e^{-\tau}\, [\tau + \frac{\tau^2}{2}]\, d\tau$$

$$x_1(t) = e^{-t}\, [4 + t + \frac{t^2}{2} - \frac{t^3}{6} - \frac{t^4}{24}] - e^{-2t}\, [4 + 4t + t^2]$$

It should be noted that $x_o(t)$ and $x_1(t)$ depend entirely on the choice of (4.4.16). Therefore, there are many other possibilities to get a first and a second approximation $x_o(t)$ and $x_1(t)$.

CHAPTER V

CONCLUSIONS

The stability of time-varying nonlinear systems has been studied. Generally, systems are described by differential equations, usually not a very convenient form for stability investigations. Consequently, an equivalent integral equation was derived using the impulse response.

The impulse response, or one-sided Green's function was shown to be the solution of a linear homogeneous differential equation subject to certain initial conditions. Based on this definition, the existence and uniqueness of the impulse response followed from customary theorems of linear differential equations. Closed form solutions could be derived by substituting special initial conditions into the general solution of a linear differential equation. In addition, if the impulse response is to be determined on either a digital or an analog computer, it is simply required to solve a linear homogeneous differential equation subject to certain initial conditions.

Several manipulations were applied to the integral equation in order to determine the stability of the appropriate solution. A procedure, common for every example, consisted in comparing two integral equations, as formulated in Theorem I.

In general, the stability condition, derived from the integral equation, was only sufficient and not necessary. In order to

obtain good results, it was found advisable to avoid approximations when possible in the corresponding derivation. In one example, this rule was strictly applied which resulted in a necessary and sufficient stability condition. The tedious solution of this example was due to the fact that almost no approximation was made to simplify the computations. From this point of view it appears that methods which yield necessary and sufficient stability conditions, will remain limited to special classes of systems.

As to the practical use of the methods discussed in this thesis, the following can be stated:

Suppose, a particular system is completely defined and it is required to determine necessary and sufficient stability conditions only. In this case, one will likely resort to a computer, conduct a number of trial and error tests, and draw a stability chart in some parameter plane.

However, if the problem is to design a system, the first preference has to be given to satisfying a certain performance criterion. According to this requirement, the control law is determined and the parameters are adjusted. Only then, one has to check whether the system exhibits 'sufficient' stability with respect to certain disturbances and parameter fluctuations. For this purpose, some of the methods, discussed in this thesis, may be adequate.

From the integral equation, a sequence of successive approximations was generated. It could be proved that this sequence was

convergent under weak restrictions. Nevertheless, it should be pointed out here that computational difficulties often limit the evaluation of this series to a few initial terms. Consequently, this type of approximation is accurate on a 'short' time interval only.

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